

# HEAT KERNEL RENORMALIZATION ON MANIFOLDS WITH BOUNDARY

BENJAMIN I. ALBERT

ABSTRACT. In the monograph *Renormalization and Effective Field Theory*, Costello gave an inductive position space renormalization procedure for constructing an effective field theory that is based on heat kernel regularization of the propagator. In this paper, we extend Costello's renormalization procedure to a class of manifolds with boundary. In addition, we reorganize the presentation of the preexisting material, filling in details and strengthening the results.

## CONTENTS

1. Introduction	1
2. Feynman Diagram Expansion	5
2.1. Feynman Diagram Expansion in the Ungraded Case	6
3. Wick's Theorem	9
3.1. Wick's Theorem on $\mathbb{R}$	9
3.2. Wick's Theorem on $(a, b)$	9
3.3. Wick's theorem on $\mathbb{R}^+$	10
3.4. Generalized Wick's Theorem on $(a, b)$	11
3.5. Wick's theorem in several variables	11
3.6. Wick's Theorem for Polytopes	12
3.7. Generalized Wick's Theorem for Compact Polytopes	13
4. Heat Kernel Counter Terms	14
4.1. Counterterms on $\mathbb{R}^n$ : Preliminaries	18
4.2. Counterterms on $\mathbb{R}^n$ : Error Bounds and Iteration	23
4.3. Counterterms on the Euclidean Upper Half Space	27
4.4. Counterterms on a Compact Manifold	32
4.5. Counterterms on a Compact Manifold with Boundary	34
5. Construction of an Effective Field Theory from a Local Functional	35
6. Some Examples	36
References	39

## 1. INTRODUCTION

Effective field theory, in the context of the renormalization group, was developed by Wilson [4] [5] based on earlier work of Kadanoff [2]. There are many variations, but the basic procedure involves two steps: mode elimination and rescaling [3] [1]. In this introduction, we shall present the intuitive idea of mode elimination and how it relates to the body of the paper.

Suppose that we have an action functional  $S[\Lambda_H](\phi)$  describing physics below an energy scale  $\Lambda_H$ . Then the action functional  $S[\Lambda_L](\phi)$  describing physics at a lower energy scale should be given by “eliminating the modes” with energy between  $\Lambda_L$  and  $\Lambda_H$ . This is described by the renormalization group equation (RGE)

$$(1) \quad e^{S[\Lambda_L](\phi)/\hbar} = \int_{\phi' \in \mathcal{F}_{(\Lambda_L, \Lambda_H)}} e^{S[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'.$$

where the integral is over  $\mathcal{F}_{(\Lambda_L, \Lambda_H)}$ , the space of fields with energy between  $\Lambda_L$  and  $\Lambda_H$ . And  $S[\Lambda](\phi)$  is defined on the low energy fields  $\phi \in \mathcal{F}_{[0, \Lambda_L]}$ . Equivalently, we can write

$$(2) \quad S[\Lambda_L](\phi) = \hbar \log \int_{\phi' \in \mathcal{F}_{(\Lambda_L, \Lambda_H)}} e^{S[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'.$$

In order to define the effective action  $S[\Lambda](\phi)$ , one might be tempted to let  $\Lambda \rightarrow \infty$  and write

$$(3) \quad S[\Lambda](\phi) = \hbar \log \int_{\phi' \in \mathcal{F}_{(\Lambda, \infty)}} e^{S(\phi+\phi')/\hbar} \mathcal{D}\phi'.$$

but this limit will not exist due to ultraviolet divergences. However, as we will see later the limit defining  $S[\Lambda](\phi)$ , will exist after an appropriate renormalization. We will continue to work with this formal expression in this introduction to motivate some of the concepts.

The focus of this paper will be on defining the effective field theory, that is making sense of (3) through an appropriate renormalization procedure.

Assume that the action is of the form

$$(4) \quad S(\phi) = -\frac{1}{2} \langle \phi, D\phi \rangle + I(\phi).$$

where  $\langle \phi, D\phi \rangle = \int_M \phi D\phi$  is the quadratic part of the action. Then because  $\phi \in \mathcal{F}_{[0, \Lambda]}$  and  $\phi' \in \mathcal{F}_{(\Lambda, \infty)}$  are orthogonal,

$$(5) \quad S(\phi + \phi') = -\frac{1}{2} \langle \phi, D\phi \rangle - \frac{1}{2} \langle \phi', D\phi' \rangle + I(\phi + \phi').$$

If  $S[\Lambda](\phi) = -\frac{1}{2} \langle \phi, D\phi \rangle + I[\Lambda](\phi)$ , then the formal expression (3) simplifies to

$$(6) \quad e^{I[\Lambda_L](\phi)/\hbar} = \int_{\phi' \in \mathcal{F}_{(\Lambda_L, \infty)}} e^{-\frac{1}{2} \langle \phi', D\phi' \rangle / \hbar + I(\phi+\phi')/\hbar} \mathcal{D}\phi'.$$

or equivalently

$$(7) \quad I[\Lambda_L](\phi) = \hbar \log \int_{\phi' \in \mathcal{F}_{(\Lambda_L, \infty)}} e^{-\frac{1}{2} \langle \phi', D\phi' \rangle / \hbar + I(\phi+\phi')/\hbar} \mathcal{D}\phi'.$$

Let  $P$  be the inverse of the quadratic form  $\langle \phi', D\phi' \rangle$  on  $\mathcal{F}_{(\Lambda_L, \infty)}$  and let  $\partial_P$  be the second order contraction operator associated to  $P$ . Continuing with our line of formal reasoning, by Wick’s theorem, a statement that is true on finite dimensional vector spaces, the integral (6) should be equal to the Wick contraction

$$(8) \quad V(P, I) = e^{\hbar \partial_P} e^{I/\hbar}.$$

and (7) is equal to the expression

$$(9) \quad W(P, I) = \hbar \log [e^{\hbar \partial_P} e^{I/\hbar}].$$

It should be noted that  $V(P, I)$  and  $W(P, I)$  are not well-defined by these formulas because  $P$  is singular, so it is necessary to regularize  $P$  and then renormalize this expression.

While the version of effective field theory with sharp energy cutoffs described above paints an intuitive physical picture, there are disadvantages to working with it, as discussed in [1]. There is an alternative approach that comes from noticing the relationship between the uncit propagator and the heat kernel. Let  $K_t(x, y)$  be the heat kernel for  $D$ . That is

$$(10) \quad \partial_t K_t(x, y) + D_x K_t(x, y) = 0$$

and  $\lim_{t \rightarrow 0^+} \int_M K_t(x, y) \phi(y) dy = \phi(x)$ . Then if the integral

$$(11) \quad G(x, y) = \int_0^\infty K_t dt$$

exists the operator it induces provides an inverse to  $D$  on  $\mathcal{F}_{(0, \infty)}$ . That is, away from the energy zero fields.

In Costello's effective field theory, we introduce the regularized propagator

$$(12) \quad P_\epsilon^L = \int_\epsilon^L K_t dt$$

and seek to define the "scale"  $L$  effective interaction as

$$(13) \quad I[L] = \lim_{\epsilon \rightarrow 0^+} \hbar \log [\exp(\hbar \partial_{P_\epsilon^L}) \exp(I/\hbar)]$$

Now  $L$  is no longer an energy scale cutoff, but rather a length scale (or time scale) regularization parameter). Again this expression needs to be renormalized. That is, an interaction functional  $I(\epsilon)$  with counterterms for  $I$  is constructed such that

$$(14) \quad I[L] = \lim_{\epsilon \rightarrow 0^+} \hbar \log [\exp(\hbar \partial_{P_\epsilon^L}) \exp((I - I(\epsilon))/\hbar)]$$

exists.

In Section 2, we define the spaces to which the propagator  $P$  and the interaction functional belong. We define stable Feynman graphs which give a way of organizing the combinatorics of the contractions in  $V(P, I)$  and  $W(P, I)$ . Theorem 1 expresses  $V(P, I)$  as a summation over all stable graphs while Corollary 1 expresses  $W(P, I)$  as a summation over connected stable graphs.

In Section 3, we state and prove several variations of Wick's theorem. In 3.2, we calculate the 1 dimensional Gaussian integral

$$(15) \quad I_{m, \alpha}(a, b) = \int_a^b x^m e^{-\alpha x^2/2} dx$$

in terms of  $I_{0, \alpha}(a, b)$  and  $J_{i, \alpha}(a, b) = x^i e^{-\alpha x^2/2} \big|_{x=a}^{x=b}$  for  $i < m$ . The formula reduces to expected results on  $\mathbb{R}$  and  $\mathbb{R}^+$  which are recalled in 3.1 and 3.3 respectively. In 3.4, we generalize the formula for  $I_{m, \alpha}(a, b)$  to one for

$$(16) \quad I_{m, \alpha, \beta}(a, b) = \int_a^b x^m e^{-\alpha x^2/2 + \beta x} dx.$$

The proof, which is analogous to the one in 3.2 is omitted. The next two sections are focused on the many variables Wick's theorem. That is, the computation of the

integral

$$(17) \quad \int_P x_{m_1} \dots x_{m_k} e^{-Q(x)/2} dx$$

where  $Q(x)$  is a nondegenerate quadratic form. In 3.5, we recall the standard statement of Wick's theorem on  $P = \mathbb{R}^n$  and give a proof by diagonalizing the quadratic form and applying the result of 3.1. This will be used to calculate the counterterms on  $\mathbb{R}^n$  in 4.1. In 3.6 it is shown that the result of 3.2 is sufficient to compute (17) inductively, when  $P$  is any polytope. Lastly, we show that as long as  $P$  is bounded the  $Q(x)$  may be degenerate and even inhomogeneous. In this case, the result of 3.4 can be applied iteratively to compute the answer. We specialize to the case relevant for the counterterms on  $\mathbb{H}^n$ , the upper half space with the Euclidean metric, in 4.3.

Section 4 forms the body of the paper. The renormalization procedure is based on the ability to cover  $(0, \infty)^k$  and a fortiori  $(\epsilon, 1)^k$  by sets defined by inequalities of the form  $t_i \leq t_j^R$ , where  $R \geq 1$ . In the introduction to Section 4, the covering lemma that was proved by Costello in [1] is strengthened and proved. Much more detail about the nature of the sets in the cover is given. Other preliminary concepts needed for the renormalization procedure like local functionals and the form of their Feynman weights are then discussed.

In 4.1, we formulate Costello's renormalization procedure on  $\mathbb{R}^n$ . We give explicit formulas whenever possible and fill in a few steps in the argument omitted by Costello, such as the introduction of what we call spanning tree coordinates. In 4.2, we show how to control the error and how the basic result of 4.1 can be used inductively to provide counterterms on each of sets in the cover of  $(\epsilon, 1)^k$  where  $k$  is the number of edges in the Feynman graph whose weight we are trying to renormalize.

In 4.3, the renormalization is adapted to  $\mathbb{H}^n$ , the upper half space with the Euclidean metric. The procedure does not carry over without modification since the quadratic form in the integral computing the Feynman weight is both no longer non-degenerate and no longer homogeneous. Luckily, this difficulty can be circumvented by a clever change of coordinates in the direction normal to boundary. The counterterms have a more complicated form than those on  $\mathbb{R}^n$ , but we argue that the inductive procedure of 4.2 can be carried out with appropriate modifications.

In 4.4, we correct what seems to be an oversight in Costello's reasoning in [1]. On a compact manifold  $M$ , Costello uses the asymptotic expansion of the heat kernel  $K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y) t^i$ , but for each chart in a cover replaces  $d(x, y)$  with the coordinate distance  $\|x - y\|$ . Thus, taking a partition of unity, the Feynman weight under consideration becomes a sum of integrals whose integrands will contain the exponential of a quadratic form, which allows us to apply Wick's theorem. However, it does not seem to be correct that  $K_t(x, y) \sim e^{-\|x - y\|^2/4t} \sum_i \phi_i(x, y) t^i$ , at least not uniformly in  $x$  and  $y$ . Again, we show how this difficulty is not fatal. While the counterterms will not simplify as they do on  $\mathbb{R}^n$ , through the introduction of spanning tree coordinates, one can still bound the error. The inductive step in the construction thus remains valid.

The culmination of these results is 4.5 where we show the renormalization procedure can be carried out on a class of compact manifolds with boundary where the argument reduces that of 4.3 near the boundary and 4.4 away from the boundary.

Lastly, in Section 5 we move beyond the construction of counterterms for each Feynman weight and construct the counterterms  $I^{CT}(\epsilon)$  for the entire effective interaction. Section 6 gives two examples of the construction of the counterterms for Feynman weights associated to 1-loop graphs in the  $\phi_4^4$ -theory.

## 2. FEYNMAN DIAGRAM EXPANSION

Let  $\mathcal{E}$  be a graded object in an appropriate symmetric monoidal category, which contains a field  $\mathbb{K}$  as its monoidal unit. For toy examples one can work with category of finite dimensional vector spaces. For quantum field theory one will need to work with a category of topological vector spaces like the category of nuclear spaces with the projective tensor product. The identifications  $(\mathcal{E} \otimes \mathcal{F})^* \cong \mathcal{E}^* \otimes \mathcal{F}^*$  and  $\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F}$  will be made throughout. We will not dwell on the issue any further.

Fix an element  $P \in \text{Sym}^2(\mathcal{E})$  which will be called a *propagator*. We define the algebra of formal power series on  $\mathcal{E}$ ,

$$(18) \quad \mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \text{Hom}(\otimes^n \mathcal{E}, \mathbb{K})_{S_n} = \prod_{n \geq 0} \text{Sym}^n(\mathcal{E}^*)$$

Here  $\text{Sym}$  means taking coinvariants of the  $n$ -fold tensor product with respect to the symmetric group action. An element of  $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$  is of the form  $I = \sum_{i,k \geq 0} I_{i,k} \hbar^i$ , where  $I_{i,k} \in \text{Sym}^k(\mathcal{E}, \mathbb{R})$ . Let

$$(19) \quad \mathcal{O}(\mathcal{E})^+[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]$$

be the functionals of the form  $I = \sum_{i,k \geq 0} I_{i,k} \hbar^i$ , where  $I_{0,k} = 0$  for  $k < 3$  and  $I_{1,0} = 0$ . We will see the reason for this restricted class of functionals later in the section.

We are interested in combinatorial formulas for “functional integrals” of the form

$$(20) \quad V(P, I) = e^{\hbar \partial_P} e^{I/\hbar}$$

and

$$(21) \quad W(P, I) = \hbar \log(e^{\hbar \partial_P} e^{I/\hbar}),$$

where  $\partial_P$  denotes the contraction operator  $\frac{1}{2} \sum_i \partial_{P_i^{(1)}} \partial_{P_i^{(2)}}$  where  $P = \sum_i P_i^{(1)} \otimes P_i^{(2)}$ .

**Lemma 1** (Feynman Expansion).

$$(22) \quad V(P, I) = \sum_{\{n_{i,k}\}} \sum_j C(\{n_{i,k}\}, j) \hbar^{p(\{n_{i,k}\}, j)} \partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}}$$

where

$$C(\{n_{i,k}\}, j) = \frac{1}{j!} \prod_{i,k} \frac{1}{n_{i,k}!}$$

and

$$p(\{n_{i,k}\}, j) = \sum_{i,k} i n_{i,k} - \sum_{i,k} n_{i,k} + j.$$

In the outer summation, we sum over the collection of double sequences of non-negative integers  $\{n_{i,k}\}_{i,k \geq 0}$  with the requirement that for all but finitely many  $i, k$ ,  $n_{i,k} = 0$ .

*Proof.* By the multinomial formula

$$\begin{aligned} \exp \left( \sum_{i,k} I_{i,k} \hbar^{i-1} \right) &= \sum_j \frac{(\sum_{i,k} I_{i,k} \hbar^{i-1})^j}{j!} \\ &= \sum_j \sum_{i,k} \prod \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}}, \end{aligned}$$

where the inner sum is over sequences of nonnegative numbers  $\{n_{i,k}\}$  such that  $\sum_{i,k} n_{i,k} = j$ . We can reexpress this as a single sum over sequences of almost all zero nonnegative integers  $\{n_{i,k}\}$

$$\exp \left( \sum_{i,k} I_{i,k} \hbar^{i-1} \right) = \sum_{\{n_{i,k}\}} \prod_{i,k} \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}}.$$

Thus,

$$\begin{aligned} V(P, I) &= \sum_{\{n_{i,k}\}} \sum_j \frac{\hbar^j}{j!} \partial_P^j \prod_{i,k} \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}} \\ &= \sum_{\{n_{i,k}\}} \sum_j C(\{n_{i,k}\}, j) \hbar^{p(\{n_{i,k}\}, j)} \partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}} \end{aligned}$$

□

It remains to investigate the combinatorial structure of the expression

$$\partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}}.$$

Before doing so, we shall make a definition.

**Definition 1.** A stable graph is defined by

$V(\gamma)$ : a set of vertices

$E(\gamma)$ : a set of edges each connecting two vertices

$T(\gamma)$ : a set of tails each connected to a single vertex

and a function  $g : V(\gamma) \rightarrow \mathbb{Z}^{\geq 0}$  associating a “genus” to each vertex.

There is a natural preorder on vertices: If  $v_1$  has  $g(v_1) = i_1$  and valency  $k_1$ , and  $v_2$  has  $g(v_2) = i_2$  and valency  $k_2$ , then  $v_1 \preceq v_2$  if  $i_1 < i_2$  or  $i_1 = i_2$  and  $k_1 \leq k_2$ .

**2.1. Feynman Diagram Expansion in the Ungraded Case.** Begin with the expression

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \left( \frac{1}{j! 2^j} \prod_{i,k} \frac{1}{n_{i,k}!} \right) \hbar^{p(\{n_{i,k}\}, j)} \left( \sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}} \right)^j \prod_{i,k} I_{i,k}^{n_{i,k}}.$$

Let  $I_{i_1, k_1}, \dots, I_{i_n, k_n}$  be the sequence of interactions for which  $n_{i,k} \neq 0$ . Recall that the propagator  $P \in \text{Sym}^2 \mathcal{E}$  with  $P = \sum_l P_l^{(1)} \otimes P_l^{(2)}$ , and we are assuming that

$\mathcal{E}$  is ungraded. Make the substitution  $I_{i,k} = S^k I_{i,k}/k!$  where  $S^k I_{i,k} = \sum_{\sigma \in S_k} I_{i,k}^\sigma = k! I_{i,k}$ . Then

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \left( \frac{1}{j! 2^j} \prod_{i,k} \frac{1}{n_{i,k}! (k!)^{n_{i,k}}} \right) \hbar^{p(\{n_{i,k}\}, j)} \left( \sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}} \right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.$$

Then

$$\left( \sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}} \right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.$$

will be a sum over contractions that can be parametrized by injections  $Q : H \rightarrow V$  of the set  $H = \{1^{(1)}, 1^{(2)}, \dots, j^{(1)}, j^{(2)}\}$  into the set of inputs to the interactions  $V = \{1^{(1)}, \dots, k_1^{(1)}, \dots, 1^{(n)}, \dots, k_n^{(n)}\}$ .

Since  $\mathcal{E}$  is ungraded, we can reorder the contractions so that the images of the index (1) elements in  $H$ ,  $Q(1^{(1)}), \dots, Q(j^{(1)})$  are in ascending order. There are  $j!$  contractions that will be reordered to the same contraction in this way. We can also reorder so that  $Q(\alpha^{(1)})$  comes before  $Q(\alpha^{(2)})$ . There are  $2^j$  contractions that will be reordered to the same contraction in this way.

Injectons up to these reorderings are in one-to-one correspondence with partitions of  $V$  into  $j$  subsets with two elements and 1 additional subset containing the remaining  $|V| - 2j$  elements. Let  $\mathcal{Q}(\{n_{i,k}\}, j)$  be the collection of such partitions and for  $Q \in \mathcal{Q}(\{n_{i,k}\}, j)$  let  $w_Q(P, I)$  denote the corresponding contraction.

Then

$$(23) \quad V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \sum_{Q \in \mathcal{Q}(\{n_{i,k}\}, j)} \left( \prod_{i,k} \frac{1}{n_{i,k}! (k!)^{n_{i,k}}} \right) \hbar^{p(\{n_{i,k}\}, j)} w_Q(P, I)$$

Any partition  $Q \in \mathcal{Q}(\{n_{i,k}\}, j)$  determines a stable graph  $\gamma$  in an obvious way. Consider  $\mathcal{Q}_\gamma(\{n_{i,k}\}, j)$ , the collection of partitions which determine the same stable graph  $\gamma$ . Let  $G(\{n_{i,k}\}, j) = \prod_{i,k} (S_k^{n_{i,k}} \rtimes S_{n_{i,k}})$ . Note that

$$|G(\{n_{i,k}\}, j)| = \prod_{i,k} n_{i,k}! (k!)^{n_{i,k}}$$

This acts on  $V$  by permuting the interactions of type  $i, k$  and their  $k$  inputs. As a consequence, it acts on  $\mathcal{Q}(\{n_{i,k}\}, j)$ . In fact, it acts transitively on  $\mathcal{Q}_\gamma(\{n_{i,k}\}, j)$ . The stabilizer subgroup of a given partition  $Q \in \mathcal{Q}_\gamma(\{n_{i,k}\}, j)$  is equal to  $\text{Aut}(\gamma)$ , the group of automorphisms of the stable graph  $\gamma$ . By the orbit-stabilizer theorem, the number of partitions which determine the same stable graph  $\gamma$  is given by

$$(24) \quad \frac{|G(\{n_{i,k}\}, j)|}{|\text{Aut}(\gamma)|} = \frac{\prod_{i,k} n_{i,k}! (k!)^{n_{i,k}}}{|\text{Aut}(\gamma)|}$$

Therefore,

**Theorem 1** (Feynman Diagram Expansion, Ungraded Case). *For a stable graph  $\gamma$ , we define*

$$(25) \quad g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(v)$$

where  $b(\gamma)$  is the first Betti number of  $\gamma$ . Let  $C(\gamma)$  be the number of connected components of  $\gamma$ . Then

$$(26) \quad V(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)-C(\gamma)} w_{\gamma}(P, I)$$

*Proof.* The constant  $p(\{n_{i,k}\}, j) = \sum_{i,k} i n_{i,k} - \sum_{i,k} n_{i,k} + j$  has a very simple interpretation in terms of the stable graph  $\gamma$  since

$$\sum_{v \in V(\gamma)} g(v) = \sum_{i,k} i n_{i,k},$$

$|V(\gamma)| = \sum_{i,k} n_{i,k}$  and  $|E(\gamma)| = j$ . Using the fact that

$$(27) \quad b(\gamma) = |E(\gamma)| - |V(\gamma)| + C(\gamma),$$

and the definition

$$(28) \quad g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(v)$$

we have

$$(29) \quad p(\{n_{i,k}\}, j) = g(\gamma) - C(\gamma).$$

Lastly define  $w_{\gamma}(P, I)$  to be  $w_Q(P, I)$  where  $Q$  is any partition that determines  $\gamma$ . The formula now follows from (23) and (24).  $\square$

Now we describe a combinatorial formula for  $W(P, I) = \hbar \log(e^{\hbar \partial_P} e^{I/\hbar})$  or equivalently  $e^{W(P, I)/\hbar} = e^{\hbar \partial_P} e^{I/\hbar}$ .

**Corollary 1.**

$$(30) \quad W(P, I) = \sum_{\gamma \text{ conn}} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)} w_{\gamma}(P, I)$$

*Proof.* If  $\gamma_1 \cup \dots \cup \gamma_k$  is the disjoint union of not necessarily distinct connected stable graphs  $\gamma_1, \dots, \gamma_k$ , then it is clear that

$$g(\gamma_1 \cup \dots \cup \gamma_k) = g(\gamma_1) + \dots + g(\gamma_k)$$

$$C(\gamma_1 \cup \dots \cup \gamma_k) = C(\gamma_1) + \dots + C(\gamma_k)$$

and if  $\gamma = (\cup^{k_1} \gamma_1) \cup \dots \cup (\cup^{k_n} \gamma_n)$  where  $\gamma_1, \dots, \gamma_n$  are distinct

$$|\text{Aut}(\gamma)| = k_1! \dots k_n! |\text{Aut}(\gamma_1)|^{k_1} \dots |\text{Aut}(\gamma_n)|^{k_n}$$

Thus,

$$\begin{aligned} \exp(W(P, I)/\hbar) &= \exp\left(\sum_{\gamma \text{ conn}} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)-1} w_{\gamma}(P, I)\right) \\ &= \sum_{\{k_{\gamma}\}} \prod_{\gamma \text{ conn}} \frac{1}{|\text{Aut}(\gamma)|^{k_{\gamma}} k_{\gamma}!} \hbar^{k_{\gamma}(g(\gamma)-1)} w_{\cup k_{\gamma} \gamma}(P, I) \\ &= \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)-C(\gamma)} w_{\gamma}(P, I) \end{aligned}$$

In the second line above, for each sequence  $\{k_{\gamma}\}_{\gamma \text{ conn}}$  in the outer summation,  $k_{\gamma} = 0$  for all but finitely many  $\gamma$ , and  $k_{\gamma}$  is a nonnegative integer for all  $\gamma$ .  $\square$



**Corollary 2.** For  $I \in \mathcal{O}(\mathcal{E})^+[[\hbar]]$ ,

$$W(P, I) \in \mathcal{O}(\mathcal{E})^+[[\hbar]]$$

### 3. WICK'S THEOREM

**3.1. Wick's Theorem on  $\mathbb{R}$ .** In one variable, Wick's theorem reduces to the statement

$$(31) \quad \int_{-\infty}^{\infty} x^m e^{-\alpha x^2/2} dx = \begin{cases} \sqrt{2\pi} \frac{(2k)!}{k! 2^k} \frac{1}{\alpha^{(2k+1)/2}} & \text{if } m = 2k \\ 0 & \text{if } m = 2k + 1. \end{cases}$$

$$(32) \quad = C_m \frac{1}{\alpha^{(m+1)/2}}$$

**3.2. Wick's Theorem on  $(a, b)$ .** There are several ways of proving the formula for  $\mathbb{R}$  which one might try to adapt. The proof by integration by parts seems the best suited and is the one we develop here.

We wish to compute the integral

$$I_{m,\alpha}(a, b) = \int_a^b x^m e^{-\alpha x^2/2} dx$$

for  $-\infty \leq a \leq b \leq \infty$  and to check that the result agrees with the standard formula for  $a = -\infty$  and  $b = \infty$ . Let

$$J_{m,\alpha}(a, b) = x^m e^{-\alpha x^2/2} \Big|_{x=a}^{x=b}.$$

By integration by parts,

$$\begin{aligned} \int_a^b x^m e^{-\alpha x^2/2} dx &= \int_a^b x^{m-1} (x e^{-\alpha x^2/2}) dx \\ &= \frac{m-1}{\alpha} \int_a^b x^{m-2} e^{-\alpha x^2/2} dx - \frac{x^{m-1}}{\alpha} e^{-\alpha x^2/2} \Big|_a^b \end{aligned}$$

That is,

$$(33) \quad I_{m,\alpha}(a, b) = \frac{m-1}{\alpha} I_{m-2,\alpha}(a, b) - \frac{1}{\alpha} J_{m-1,\alpha}(a, b).$$

For  $m$  even, we can thus express  $I_{m,\alpha}(a, b)$  in terms of  $I_{0,\alpha}(a, b)$  and  $J_{l,\alpha}(a, b)$  where  $l$  ranges over odd integers less than  $m$ . For  $m$  odd, since  $I_{1,\alpha}(a, b) = -(1/\alpha) J_{0,\alpha}(a, b)$ , we can express  $I_{m,\alpha}(a, b)$  in terms of  $J_{l,\alpha}(a, b)$ , where  $l$  ranges over even integers less than  $m$ .

We can then prove a precise formula by induction:

**Proposition 1.**

$$(34) \quad I_{m,\alpha}(a, b) = \frac{C_m}{\alpha^{m/2}} I_{0,\alpha}(a, b) - \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} J_{m-1-2i,\alpha}(a, b),$$

where  $C_m = 0$  when  $m$  is odd and  $C_m = (m-1)!!$  when  $m$  is even and for all  $m$

$$(35) \quad \tilde{C}_{i,m} = \frac{(m-1)!!}{(m-1-2i)!!}$$

*Proof.* The even and odd base cases when  $m = 0$  and  $m = 1$  are clearly satisfied. Suppose the result is true for  $I_{m,\alpha}(a, b)$ . Then using (33),

$$\begin{aligned} I_{m+2,\alpha}(a, b) &= \frac{m+1}{\alpha} \frac{C_m}{\alpha^{m/2}} - \frac{m+1}{\alpha} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} J_{m-1-2i,\alpha}(a, b) \\ &\quad - \frac{1}{\alpha} J_{m+1,\alpha}(a, b) \end{aligned}$$

and

$$\begin{aligned} (m+1)\tilde{C}_{i,m} &= \frac{(m+1)!!}{(m+1-2(i+1))!!} \\ &= \tilde{C}_{i+1,m+2} \end{aligned}$$

so

$$\begin{aligned} &\frac{m+1}{\alpha} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} J_{m-1-2i,\alpha}(a, b) \\ &= \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\tilde{C}_{i+1,m+2}}{\alpha^{(i+1)+1}} J_{(m+2)-1-2(i+1),\alpha}(a, b) \\ &= \sum_{i=1}^{\lfloor \frac{(m+2)+1}{2} \rfloor} \frac{\tilde{C}_{i,m+2}}{\alpha^{i+1}} J_{(m+2)-1-2i}(a, b) \end{aligned}$$

The induction step is now completed by employing the fact that

$$\frac{(m+1)C_m}{\alpha\alpha^{m/2}} = \frac{C_{m+2}}{\alpha^{(m+1)/2}}$$

□

As  $a \rightarrow -\infty$  and  $b \rightarrow \infty$ , we have  $J_{m,\alpha} \rightarrow 0$  and  $I_{0,\alpha} \rightarrow \sqrt{2\pi/\alpha}$ . Combining this with identity for the double factorial

$$(2k-1)!! = \frac{(2k)!}{k!2^k}$$

we recover the statement of Wick's theorem on  $\mathbb{R}$ .

**3.3. Wick's theorem on  $\mathbb{R}^+$ .** Note that if  $a = 0$  and  $b = \infty$ , then  $J_{l,\alpha} = 0$  for  $l \neq 0$  and  $J_{0,\alpha} = -1$ . Since  $(2k)!! = 2^k k!$ ,

$$(36) \quad \int_0^\infty x^m e^{-\alpha x^2/2} dx = \begin{cases} \frac{C_m}{\alpha^{m/2}} I_{0,\alpha}(0, \infty) & m \text{ even} \\ \frac{\tilde{C}_{m, \frac{m-1}{2}}}{\alpha^{(m+1)/2}} J_{0,\alpha}(0, \infty) & m \text{ odd} \end{cases}$$

$$(37) \quad = \begin{cases} \sqrt{2\pi} \frac{(2k)!}{k!2^{k+1}} \frac{1}{\alpha^{(2k+1)/2}} & \text{if } m = 2k \\ 2^k k! \frac{1}{\alpha^{k+1}} & \text{if } m = 2k+1. \end{cases}$$

**3.4. Generalized Wick's Theorem on  $(a, b)$ .** In 4.3 we shall encounter integrals of polynomials with respect to inhomogeneous quadratic forms. Here we establish the one dimensional result that can be used iteratively to calculate such integrals explicitly.

We wish to compute the integral

$$I_{m,\alpha,\beta}(a, b) = \int_a^b x^m e^{-\alpha x^2/2 + \beta x} dx$$

for  $-\infty \leq a \leq b \leq \infty$  and to check that the result agrees with the standard formula for  $a = -\infty$  and  $b = \infty$ . Let

$$J_{m,\alpha,\beta}(a, b) = x^m e^{-\alpha x^2/2 + \beta x} \Big|_{x=a}^{x=b}.$$

Firstly, (33) generalizes to

$$(38) \quad I_{m,\alpha,\beta}(a, b) = -\frac{1}{\alpha} J_{m-1,\alpha,\beta}(a, b) + \frac{\beta}{\alpha} I_{m-1,\alpha}(a, b) + \frac{m-1}{\alpha} I_{m-2,\alpha,\beta}(a, b).$$

The following is a generalization of Proposition 1

**Proposition 2.**

$$(39) \quad I_{m,\alpha,\beta}(a, b) = - \sum_{i=0}^{m-1} \sum_{\substack{\{a_j\} \\ \sum a_j = i}} \frac{\beta^{|a^{-1}(1)|} \prod_{k \in a^{-1}(2)} (s_k - 1 + m - i)}{\alpha^{|l(a)|+1}} J_{\alpha,\beta,m-i-1}$$

$$(40) \quad + \sum_{\substack{\{a_j\} \\ \sum a_j = m}} \frac{\beta^{|a^{-1}(1)|} \prod_{k \in a^{-1}(2)} (s_k - 1)}{\alpha^{|l(a)|}} I_{\alpha,\beta,0}$$

where  $\{a_j\}$  ranges over finite sequences such that  $a_j \in \{1, 2\}$  for all  $j$ . We use  $l(a)$  to denote the length of the sequence  $\{a_i\}$  and  $s_i = \sum_{j=1}^{l(a)} a_j$ .

We shall not give the proof which is a straightforward induction like the proof of Proposition 1. However, let us just check that it reduces to the formula of Proposition 1 in the case that  $\beta = 0$ . Since  $0^0 = 1$  and  $0^k = 0$  for  $k > 0$  the only nonzero terms in the sums will come from sequences with  $a^{-1}(1) = \emptyset$ . But there is exactly one such sequence such that  $\sum a_j = i$  for  $i$  even and it has  $l(a) = i/2$  and no such sequences for  $i$  odd. It is clear that this then becomes the formula of Proposition 1.

**3.5. Wick's theorem in several variables.** Suppose that  $A$  is an invertible symmetric  $n \times n$  matrix and consider the associated quadratic form  $Q(x) = \langle x, Ax \rangle = x^i A_{ij} x^j$ . We wish to compute the integral

$$I_{J,A} = \int_P x_{m_1} \dots x_{m_k} e^{-Q(x)/2} dx$$

where  $J = (j_1, \dots, j_n)$  is a multi-index such that  $x_1^{j_1} \dots x_n^{j_n} = x_{m_1} \dots x_{m_k}$  and  $P$  is a polytope.

**Theorem 2** (Wick's Theorem on  $\mathbb{R}^n$ ). *For  $k$  even*

$$(41) \quad \int_{\mathbb{R}^n} x_{m_1} \dots x_{m_k} e^{-Q(x)/2} dx = \frac{\sqrt{2\pi}}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} A_{m_{\beta_j^{(1)}}, m_{\beta_j^{(2)}}}^{-1}$$

where the sum is over partitions of the set  $1, \dots, k$  into  $k/2$  subsets of 2 elements. Here  $\beta_j^{(1)}$  and  $\beta_j^{(2)}$  denote respectively the first and second elements of the  $j$ -th set in the partition.

*Proof.* Let  $D$  denote the diagonalization of  $A$  and assume that  $D$  has diagonal entries  $\alpha_1, \dots, \alpha_n$ . In this new basis, using the change of basis matrix  $S$ , we have a linear combination

$$\sum_{i_1, \dots, i_k} S_{m_1}^{i_1} \dots S_{m_k}^{i_k} \int_{\mathbb{R}^n} y_{i_1} \dots y_{i_k} e^{-\alpha_1 x^2/2} \dots e^{-\alpha_n x^2/2} dx.$$

Apply Wick's theorem on  $\mathbb{R}$  separately in each variable. For each such integral, this gives

$$\begin{aligned} \frac{1}{\sqrt{\alpha_1 \dots \alpha_n}} \prod_{i=1}^n \frac{C_{k_i}}{\alpha_i^{k_i}} &= \frac{1}{\sqrt{\det(A)}} \prod_{i=1}^n \frac{C_{k_i}}{D_{ii}^{k_i}} \\ &= \frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} D_{i_{\beta_j^{(1)}}, i_{\beta_j^{(2)}}}^{-1} \end{aligned}$$

where the sum is over partitions of the set  $1, \dots, k$  into  $k/2$  subsets of 2 elements. We then switch the order of summation so that the sum over partitions is the outer sum and then

$$\frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \sum_{i_1, \dots, i_k} S_{m_1}^{i_1} \dots S_{m_k}^{i_k} \prod_{j=1}^{k/2} D_{i_{\beta_j^{(1)}}, i_{\beta_j^{(2)}}}^{-1} = \frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} A_{m_{\beta_j^{(1)}}, m_{\beta_j^{(2)}}}^{-1}$$

□

**3.6. Wick's Theorem for Polytopes.** The purpose of this section is to show that the result of 3.2 can be used inductively to calculate a Wick integral over a polytope in  $\mathbb{R}^n$ .

By the spectral theorem,  $A$  can be diagonalized by an orthogonal transformation. This will produce a linear combination of integrals of the form

$$\int_P x_1^{k_1} \dots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_n x_n^2/2} dx.$$

where  $P$  is some polytope.

Decompose the integral as

$$\int_{P'} \int_{a_n}^{b_n} x_1^{k_1} \dots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_n x_n^2/2} dx_n dx'.$$

where  $P'$  is the projection of  $P$  onto the hyperplane  $x_n = 0$  and  $a_n$  and  $b_n$  are piecewise linear in the variables  $x_1, \dots, x_{n-1}$ . The subdomains where  $a_n$  is linear are the projections of the  $(n-1)$ -dimensional faces of  $P$  onto the hyperplane  $x_n = 0$ . Denote an arbitrary projection of an  $(n-1)$ -dimensional face by  $P^a$ . Similarly, use  $P^b$  for an arbitrary subdomain where  $b_n$  is linear.

We apply Wick's theorem in one variable to  $x_n$  to get a linear combination of elements of the form

$$(42) \quad \int_{P'} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} J_{m-2i-1, \alpha_n}(a_n, b_n) dx'$$

and an element of the form

$$\int_{P'} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} I_{0, \alpha_n}(a_n, b_n) dx.$$

By substituting the definition of  $J_{m-2i-1, \alpha}$ , terms of the first form are equal to the summation

$$\begin{aligned} & \sum_{P^b} \int_{P^b} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} b_n^{k_n-2i-1} e^{-\alpha_n b_n^2/2} dx' \\ & - \sum_{P^a} \int_{P^a} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} a_n^{k_n-2i-1} e^{-\alpha_n a_n^2/2} dx' \end{aligned}$$

We emphasize that  $b_n|_{P^b}$  is a linear function in the variables  $x_1, \dots, x_{n-1}$  and similarly for  $a_n|_{P^a}$ .

Let us focus our attention on any term involving  $b_n|_{P^b} = d_1 x_1 + \dots + d_{n-1} x_{n-1}$ ; that is, those in the first summation. The analysis for terms involving  $a_n|_{P^a}$  in the second summation is similar.

Since  $P^b$  is a polytope, to complete the inductive step, it suffices to show that

$$(43) \quad \alpha_1 x_1^2 + \dots + \alpha_{n-1} x_{n-1}^2 + \alpha_n (d_1 x_1 + \dots + d_{n-1} x_{n-1})^2$$

is nondegenerate. Let  $d$  be the thought of as a column vector. Let  $c = \sqrt{\alpha_n} d$  and let  $A = \text{diag}(\alpha_1, \dots, \alpha_{n-1})$ . Then

$$\begin{aligned} \det(A + \alpha_n d d^t) &= \det(A) \det(I + A^{-1} c c^t) \\ &= \det(A) (1 + c^t A^{-1} c) = \det(A) (1 + |\sqrt{A^{-1}} c|^2) > 0 \end{aligned}$$

which implies that the quadratic form is nondegenerate.

**3.7. Generalized Wick's Theorem for Compact Polytopes.** In Section 4.3, the counterterms that will be introduced to renormalize the theory on  $\mathbb{H}^n$  will involve integrals of the form

$$(44) \quad \int_{P_u} e^{-Q^{(P)}(z, u, \mathbf{t})} z^{K'} dz$$

where  $P_u$  is given by the inequalities

$$\begin{aligned} 0 &\leq u + z_1 \\ 0 &\leq u + z_2 - z_1 \\ &\dots \\ 0 &\leq u + z_{m-1} - z_{m-2} \\ 0 &\leq u - z_{m-1}. \end{aligned}$$

and  $Q^{(P)}(z, u, \mathbf{t}) = \sum_{i,j} a_{ij} z_i z_j + \sum_i b_i u z_i$

We reexpress the above inequalities in a form which makes it possible to calculate the integral.

$$\begin{aligned} -u &\leq z_1 \leq (m-1)u \\ z_1 - u &\leq z_2 \leq (m-2)u \\ &\dots \\ z_{m-3} - u &\leq z_{m-2} \leq 2u \\ z_{m-2} - u &\leq z_{m-1} \leq u \end{aligned}$$

So for  $z^{K'} = z_1^{p_1} \dots z_{m-1}^{p_{m-1}}$ , we have  $\int_{P_u} e^{-Q^{(P)}(z,u,\mathbf{t})} z^{K'} dz$  is equal to

$$(45) \quad \int_{-u}^{(m-1)u} \dots \int_{z_{m-2}-u}^u e^{-\sum_{i,j} a_{ij} z_i z_j - \sum_i b_i u z_i} z_1^{p_1} \dots z_{m-1}^{p_{m-1}} dz_{m-1} \dots dz_1.$$

Despite the quadratic form being inhomogeneous with homogeneous part not necessarily being nondegenerate, since the bounds are linear and finite we are able to inductively apply the result of 3.4.

#### 4. HEAT KERNEL COUNTER TERMS

4.0.1. *Covering*  $(0, \infty)^{|E(\gamma)|}$ . Let  $k = |E(\gamma)|$ . We denote  $\mathbf{t} = (t_1, \dots, t_k)$ . For each permutations  $\sigma \in S_k$ , there is a subset

$$(46) \quad S_\sigma = \{\mathbf{t} \in (0, \infty)^k : t_{\sigma(1)} < \dots < t_{\sigma(k)}\}.$$

and it is clear that

$$(47) \quad \cup_{\sigma \in S_k} \overline{S_\sigma} = (0, \infty)^k.$$

The procedure we are about to describe is applied separately within each of the  $S_\sigma$ , but we work within

$$S_{\text{id}} = \{\mathbf{t} \in (0, \infty)^k : t_1 < \dots < t_k\}.$$

for notational clarity. We assume that  $R > 1$ .

**Definition 2.** For  $j \in \{1, \dots, k-1\}$ , let

$$(48) \quad B_R^j = \{\mathbf{t} \in S_{\text{id}} : t_j < t_{j+1}^R\}.$$

For  $i, j \in \{1, \dots, k\}$  with  $i < j$ , define

$$(49) \quad C_R^{i,j} = \{\mathbf{t} \in S_{\text{id}} : t_j^R < t_i\}.$$

and define

$$\begin{aligned} D_R^{i,j} &= S_{\text{id}} \setminus \overline{C_R^{i,j}} \\ &= \{\mathbf{t} \in S_{\text{id}} : t_j^R > t_i\}. \end{aligned}$$

And lastly for  $j \in \{2, \dots, k-1\}$ , define

$$(50) \quad A_R^j = B_R^j \cap C_R^{1,j}$$

$$(51) \quad = \{\mathbf{t} \in S_{\text{id}} : t_j < t_{j+1}^R \text{ and } t_j^R < t_1\}$$

and let  $A_R^1 = B_R^1$  and  $A^k = C_R^{1,k}$ .

Note that  $D_R^{j,j+1} = B_R^j$ . A couple of facts about these subsets are collected in the following proposition:

**Proposition 3.** For  $i_1 < i_2 < i_3$ .

$$(52) \quad C_R^{i_1, i_2} \cap C_S^{i_2, i_3} \subset C_{RS}^{i_1, i_3}$$

and similarly

$$(53) \quad D_R^{i_1, i_2} \cap D_S^{i_2, i_3} \subset D_{RS}^{i_1, i_3}$$

*Proof.* If  $\mathbf{t} \in C_R^{j_1, j_2} \cap C_S^{j_2, j_3}$ , then  $t_{i_2}^R < t_{i_1}$  and  $t_{i_3}^S < t_{i_2}$ . This implies that

$$t_{i_3}^{RS} < t_{i_1}.$$

The proof of the second inclusion is similar.  $\square$

The following statements are trivially true:

**Proposition 4.** For  $j \in \{1, \dots, k-1\}$ , let

$$(54) \quad \tilde{B}_R^j = \{\mathbf{t} \in S_{id} : t_\alpha < t_\beta^R, \text{ for } \alpha \leq j \text{ and } j+1 \leq \beta\}.$$

For  $i, j \in \{1, \dots, k\}$  with  $i < j$ , define

$$(55) \quad \tilde{C}_R^{i, j} = \{\mathbf{t} \in S_{id} : t_\alpha^R < t_\beta, \text{ for } \alpha \leq i \text{ and } j \leq \beta\}.$$

Then  $\tilde{B}_R^j = B_R^j$  and  $\tilde{C}_R^{i, j} = C_R^{i, j}$

**Proposition 5.** For  $j_1 \leq j_2$ , if  $C_R^{i, j_1} \supseteq C_R^{i, j_2}$ .

The next two propositions are needed to prove Theorem 4.

**Proposition 6.**  $C_R^{i, j} \cap D_R^{l, m} = \emptyset$  for  $i \leq l$  and  $m \leq j$ .

*Proof.* If  $t_j^R < t_i$  and  $t_l < t_m^R$ . Then

$$t_i \leq t_l < t_m^R < t_j^R < t_i,$$

a contradiction.  $\square$

**Proposition 7.**  $B_R^l \cap C_R^{i, j} = \emptyset$  for  $i \leq l < j$ .

*Proof.* Since  $B_R^l = D_R^{l, l+1}$ , we can apply the previous proposition.  $\square$

**Definition 3.** We consider sequences of the form  $1 = i_0 < i_1 < \dots < i_m \leq k$ , where  $m \leq k-1$ . For any sequence of this form  $I$ , we define the sets  $E_R^I = \cap_{j=0}^m E_{R, j}^I$ , where  $E_{R, i}^I$  is defined such that

$$E_{R, 0}^I = \begin{cases} B_{R^{s_1}}^1 & \text{if } m = 0 \\ S_{id} & \text{otherwise} \end{cases}$$

and

$$E_{R, j}^I = C_{R^{s_j}}^{i_{j-1}, i_j} \cap D_{R^{s_j}}^{i_{j-1}, i_j+1}$$

and for  $j = m$

$$E_{R, m}^I = \begin{cases} C_{R^{s_m}}^{i_{m-1}, i_m} \cap D_{R^{s_m}}^{i_{m-1}, i_m+1} \cap B_{R^{s_{m+1}}}^{i_m} & \text{if } i_m \neq k \\ C_{R^{s_m}}^{i_{m-1}, i_m} & \text{if } i_m = k. \end{cases}$$

where  $s_0, \dots, s_m$  is a fixed sequence.

**Theorem 3.** Their closures  $\overline{E}_R^I$  form a cover of  $(0, \infty)^k$ .

*Proof.* If  $t_1 \leq t_2^{R^{s_1}}$ , then  $\mathbf{t} \in \overline{B}_R^1$ . Thus let  $m = 0$ .

Otherwise, assume let  $i_1$  be the largest integer such that  $t_{i_1}^{R^{s_1}} \leq t_1 = t_{i_0}$ . Then  $i_1 \in \overline{C}_{R^{s_1}}^{i_0, i_1}$ . If  $i_1 = k$ , let  $m = 1$ . If  $i_1 < k$ , then  $\mathbf{t} \in D_{R^{s_1}}^{i_0, i_1+1}$ . If  $t_{i_1} \leq t_{i_1+1}^R$  then  $\mathbf{t} \in \overline{B}_{R^{s_2}}^{i_1}$  and we let  $m = 1$ .

Otherwise, let  $i_2$  be the largest integer such that  $t_{i_2}^{R^{s_2}} \leq t_{i_1}$ . Then  $i_2 \in \overline{C}_{R^{s_2}}^{i_1, i_2}$ . If  $i_2 = k$ , let  $m = 2$ . If  $i_2 < k$ , then  $\mathbf{t} \in D_{R^{s_2}}^{i_1, i_2+1}$ . If  $t_{i_2} \leq t_{i_2+1}^{R^{s_3}}$  then  $\mathbf{t} \in \overline{B}_{R^{s_3}}^{i_2}$  and we let  $m = 2$ .

And so on ... □

**Theorem 4.** *The sets  $E_R^I$  are disjoint.*

*Proof.* We prove this by induction. Consider the distinct sequences  $1 = i_0 < i_1 < \dots < i_m \leq k$  and  $1 = j_0 < j_1 < \dots < j_n \leq k$ , where without loss of generality we assume that  $m \leq n$ .

Suppose that  $i_l \neq j_l$ , but  $i_1 = j_1, \dots, i_{l-1} = j_{l-1}$ . Then

$$\begin{aligned} E_{R,l}^I \cap E_{R,l}^J &\subseteq C_{R^{s_l}}^{i_{l-1}, i_l} \cap D_{R^{s_l}}^{i_{l-1}, i_l+1} \cap C_{R^{s_l}}^{i_{l-1}, j_l} \cap D_{R^{s_l}}^{i_{l-1}, j_l+1} \\ &= \emptyset. \end{aligned}$$

because  $C_R^{i,j} \cap D_R^{i,m} = \emptyset$  for  $m \leq j$  by Proposition 6.

It is also possible that  $i_1 = j_1, \dots, i_m = j_m$ , but  $m < n$ . Then

$$\begin{aligned} E_m^I \cap E_{m+1}^J &\subseteq B_{R^{s_{m+1}}}^{i_m} \cap C_{R^{s_{m+1}}}^{i_m, j_{m+1}} \\ &= \emptyset. \end{aligned}$$

by Proposition 7. □

Now specialize to a specific sequence  $s_0 = 1$  and  $s_i = 2^{i-1}$  for  $i > 0$ .

**Theorem 5.** *Consider the sequence  $1 = i_0 < i_1 < \dots < i_m \leq k$ . Then*

$$E_R^I \subseteq A_{R^{2^m}}^{i_m}$$

*Proof.* If  $m = 0$ , it is clear that  $E_R^I \subseteq A_R^1 = B_R^1$ .

If  $m > 0$ ,

$$\begin{aligned} E_R^I &\subseteq \begin{cases} C_R^{i_0, i_1} \cap C_R^{i_1, i_2} \cap C_{R^2}^{i_2, i_3} \dots \cap C_{R^{2^{m-1}}}^{i_{m-1}, i_m} \cap B_{R^{2^m}}^{i_m} & \text{if } i_m < k \\ C_R^{i_0, i_1} \cap C_R^{i_1, i_2} \cap C_{R^2}^{i_2, i_3} \dots \cap C_{R^{2^{m-1}}}^{i_{m-1}, i_m} & \text{if } i_m = k \end{cases} \\ &\subseteq \begin{cases} C_{R^{2^m}}^{i_0, i_m} \cap B_{R^{2^m}}^{i_m} & \text{if } i_m < k \\ C_{R^{2^m}}^{i_0, i_m} & \text{if } i_m = k \end{cases} \\ &= A_{R^{2^m}}^{i_m} \end{aligned}$$

□

The construction of the counterterms in 4.2.2 will be based on a refinement of the covering  $\{\overline{E}_R^I\}$ . For  $l < k$ , given a sequence  $l = i_0 < \dots < i_m \leq k$ , introduce the more general sets  $E_R^I$  which are defined by applying the definition of  $E_R^I$ , but replacing the set  $\{t_1 < \dots < t_k\}$  with the set  $\{t_l < \dots < t_k\}$ . For  $l = 1$ , we recover  $E_R^I$  in the sense in which it was defined earlier.

The following is a corollary of Theorem 3:



**Corollary 3.** *Consider the collection of sequences of the form*

$$\begin{aligned} 1 &= i_0^{(1)} < i_1^{(1)} < \dots < i_{m^{(1)}}^{(1)} \\ i_{m^{(1)}}^{(1)} &= i_0^{(2)} < i_1^{(2)} < \dots < i_{m^{(2)}}^{(2)} \\ &\dots \\ i_{m^{(p-1)}}^{(p-1)} &= i_0^{(p)} < i_1^{(p)} < \dots < i_{m^{(p)}}^{(p)} = k. \end{aligned}$$

Then the sets

$$(56) \quad \overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} \dots \cap \overline{E}_R^{I^{(p)}}$$

form a cover of  $S_{id}$ .

**4.0.2. Differential Operators.** Let  $M$  be a smooth manifold, let  $E$  be a graded vector bundle and let  $\underline{\mathbb{R}}$  be the trivial line bundle. Let  $\mathcal{E} = \Gamma(E)$  and  $C^\infty(M) = \Gamma(\underline{\mathbb{R}})$ . A differential operator  $P : E \rightarrow \underline{\mathbb{R}}$  is an  $\mathbb{R}$ -linear map  $\mathcal{E} \rightarrow C^\infty(M)$  which can be given locally using Einstein notation

$$s = \alpha^i e_i \mapsto a_j^I \frac{\partial \alpha^j}{\partial x^I}$$

where  $e_1, \dots, e_r$  is a local (homogeneous) frame for  $E$  on some sufficiently small coordinate neighborhood  $U$ , and  $\alpha^1, \dots, \alpha^r$  and  $a_i^I$  are functions on  $U$ .

Equivalently, there is a bundle map  $\iota_P : J(E) \rightarrow \underline{\mathbb{R}}$ , where  $J(E)$  is the jet bundle of  $E$ . The differential operator  $P$  is determined by  $\iota_P$  by composing with the jet prolongation of  $s$ ,  $j(s) : M \rightarrow J(E)$ . That is,  $P(s) = \iota_P \circ j(s)$ .

**4.0.3. Local Functionals.**

**Definition 4.** A local functional  $I \in \mathcal{O}_{loc}^k(\mathcal{E})$  of degree  $k$  is a functional  $I \in \mathcal{O}^k(\mathcal{E})$  of the form

$$(57) \quad I(s) = \sum_{\beta=1}^m \int_M D_{\beta,1}(s) \dots D_{\beta,k}(s)$$

for some collection of differential operators  $D_{i,j} : E \rightarrow \underline{\mathbb{R}}$ .

Substituting the local formula for the differential operators

$$D_{\beta,j}(\alpha^i e_i) = (a_{\beta,j})_k^I \frac{\partial \alpha^k}{\partial x^I}$$

we get that locally

$$(58) \quad I(\alpha^i e_i) = \sum_{\beta=1}^m \int_U (a_{\beta,1})_{j_{\beta,1}}^{I_{\beta,1}} \dots (a_{\beta,k})_{j_{\beta,k}}^{I_{\beta,k}} \frac{\partial \alpha^{j_{\beta,1}}}{\partial x^{I_{\beta,1}}} \dots \frac{\partial \alpha^{j_{\beta,k}}}{\partial x^{I_{\beta,k}}}$$

$$(59) \quad = \int_U a_{j_1, \dots, j_k}^{I_1, \dots, I_k} \frac{\partial \alpha^{j_1}}{\partial x^{I_1}} \dots \frac{\partial \alpha^{j_k}}{\partial x^{I_k}}.$$

for a collection of functions  $a_{j_1, \dots, j_k}^{I_1, \dots, I_k}$  on  $U$ .

4.0.4. *Evaluation of  $w_\gamma(P, I)$ .* We would like to describe the form of  $w_\gamma(P, I)$  when  $I \in \mathcal{O}_{\text{loc}}(\mathcal{E})[[\hbar]]$  is a power series of local functionals and

$$(60) \quad P = P_\epsilon^L = \int_\epsilon^L K_t dt$$

where  $K_t$  is the heat kernel of  $M$ .

We shall work in the ungraded case. For notation simplicity, we shall also assume from hereon out that  $E = \mathbb{R}$ , although the method remains valid for any vector bundle.

For each vertex  $v \in V(\gamma)$ , we associate the functional  $I_{g(v), k(v)}$ , where  $k$  is the valency of the vertex  $v$ . Assume that within a given chart  $U$ ,

$$S^{k(v)} I_{g(v), k(v)}(\alpha_1^i e_i, \dots, \alpha_{k(v)}^i e_i) = \int_U a^{I^{v^1}, \dots, I^{v^{k(v)}}} \frac{\partial \alpha_1}{\partial x^{I^{v^1}}} \cdots \frac{\partial \alpha_{k(v)}}{\partial x^{I^{v^{k(v)}}}}.$$

where  $I^{v^1}, \dots, I^{v^{k(v)}}$  ranges over multi-indices with  $|I^{v^1}| + \cdots + |I^{v^{k(v)}}| \leq \text{ord } I_{g(v), k(v)}$ . Choose an ordering on the set of half edges  $v^1, \dots, v^{k(v)}$  incident on each vertex  $v$  and an orientation on each edge. Then  $\gamma$  determines the maps

$$\begin{aligned} Q : T(\gamma) &\rightarrow \cup_{v \in V} \{v^1, \dots, v^{k(v)}\} = H(\gamma) \\ Q_1 : E(\gamma) &\rightarrow \cup_{v \in V} \{v^1, \dots, v^{k(v)}\} \\ Q_2 : E(\gamma) &\rightarrow \cup_{v \in V} \{v^1, \dots, v^{k(v)}\} \end{aligned}$$

where  $Q_1$  and  $Q_2$  map an edge to its first and second half edges respectively, and  $Q$  maps a tail to itself. Also denote by  $v_1(e)$  and  $v_2(e)$  the first and second vertices of the edge  $e$ . Similarly, let  $v(h)$  denote the vertex of the tail  $h$ .

With these data, we can give the expression

$$(61) \quad w_\gamma(P, I) = \int_{(\epsilon, L)^{|E(\gamma)|}} f_\gamma[P, I].$$

where for  $M = \mathbb{R}^n$ ,

$$(62) \quad f_\gamma[P, I] = \int_{\mathbb{R}^{n|V(\gamma)|}} \prod_{v \in V(\gamma)} a^{I^{v^1}, \dots, I^{v^{k(v)}}}(x_v) \prod_{e \in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}}$$

where  $k$  is used to stand for  $k(v)$ . If  $M$  is a compact manifold then choose a partition of unity subordinate to a finite cover of  $M$  (on which  $E$  is trivialized). Then  $f_\gamma[P, I]$  is a sum of integrals of the form

$$(63) \quad \int_{U^{|V(\gamma)|}} \chi \prod_{v \in V(\gamma)} a^{I^{v^1}, \dots, I^{v^{k(v)}}}(x_v) \prod_{e \in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}}$$

where  $\chi$  is the partition of unity function for the open set  $U$  in the cover and  $\alpha^i$  are the coordinates of  $\alpha$  in  $U$ .

Due to the symmetry of  $P$  and  $I_{i,k}$ , the value of  $w_\gamma(P, I)$  is independent of the choices of ordering and orientation.

4.1. **Counterterms on  $\mathbb{R}^n$ : Preliminaries.** When working with  $\mathbb{R}^n$ , we shall only consider scalar field theories, so that the heat kernel has the simple form

$$(64) \quad K_t(x, y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.$$

4.1.1. Derivatives of  $K_t$ .

**Proposition 8.** For a multi-index  $I = (i_1, \dots, i_n)$ ,  $\frac{\partial K_t}{\partial x^I}$  is a polynomial in  $x_1, \dots, x_n, y_1, \dots, y_n$  and  $1/t$  which is multiplied by  $K_t$ . The degree in  $1/t$  is  $|I|$ .

The proof will be a consequence of the following lemma which gives explicit formulas for the single variable derivatives.

**Lemma 2.** For the heat kernel in (64)

$$(65) \quad \frac{\partial K_t}{\partial x_i^k} = P_{i,k} K_t$$

where  $P_{i,k}$  is polynomial in  $x_i$  and  $y_i$  and  $1/t$ . The degree of  $P_{i,k}$  in  $1/t$  is  $k$ .

*Proof.* We would like to find an explicit expression for  $P_{i,k}$ .

Note that

$$\frac{\partial K_t}{\partial x_i} = \frac{x_i - y_i}{2t} K_t.$$

For each sequence of the form  $s_1, \dots, s_{k'}$  where  $s_j \geq 1$  for all  $j$  and  $s_1 + \dots + s_{k'} = k$ , consider the functions

$$F_{s_1, \dots, s_{k'}}(t, x_i, y_i) = \begin{cases} \partial_{x_i}^{s_1} \left[ \left( \frac{x_i - y_i}{2t} \right)^{s_2} \dots \partial_{x_i}^{s_{k'-1}} \left[ \left( \frac{x_i - y_i}{2t} \right)^{s_{k'}} \right] \dots \right] & k' \text{ even} \\ \left( \frac{x_i - y_i}{2t} \right)^{s_1} \partial_{x_i}^{s_2} \left[ \left( \frac{x_i - y_i}{2t} \right)^{s_3} \dots \partial_{x_i}^{s_{k'-1}} \left[ \left( \frac{x_i - y_i}{2t} \right)^{s_{k'}} \right] \dots \right] & k' \text{ odd.} \end{cases}$$

We argue by induction that

$$\frac{\partial^k K_t}{\partial x_i^k} = \sum_{\substack{s_1 + \dots + s_{k'} = k \\ s_j \geq 1 \text{ for all } j}} F_{s_1, \dots, s_{k'}} K_t.$$

For  $k = 1$ , this is clearly true. Suppose it is true for some  $k \geq 1$ , then

$$\begin{aligned} \frac{\partial^{k+1} K_t}{\partial x_i^{k+1}} &= \frac{\partial}{\partial x_i} \sum_{\substack{s_1 + \dots + s_{k'} = k \\ s_j \geq 1 \text{ for all } j}} F_{s_1, \dots, s_{k'}} K_t \\ &= \sum_{\substack{s_1 + \dots + s_{k'} = k \\ s_j \geq 1 \text{ for all } j}} \partial_{x_i} F_{s_1, \dots, s_{k'}} K_t \\ &\quad + \sum_{\substack{s_1 + \dots + s_{k'} = k \\ s_j \geq 1 \text{ for all } j}} F_{s_1, \dots, s_{k'}} \left( \frac{x_i - y_i}{2t} \right) K_t \\ &= \sum_{\substack{s_1 + \dots + s_{k'} = k+1 \\ s_j \geq 1 \text{ for all } j}} F_{s_1, \dots, s_{k'}} K_t. \end{aligned}$$

In fact, we can be more precise. That is, for  $k'$  even

$$F_{s_1, \dots, s_{k'}} = \frac{s_{k'}!}{(s_{k'} - s_{k'-1})!} \dots \frac{(s_{k'} - s_{k'-1} + \dots + s_2)!}{(s_{k'} - s_{k'-1} + \dots - s_1)!} \frac{((x_i - y_i)/2t)^{s_{k'} + s_{k'-2} + \dots + s_2}}{(x_i - y_i)^{s_{k'-1} + s_{k'-3} + \dots + s_1}}$$

as long as  $s_{k'} - s_{k'-1} + \dots + s_{2i} - s_{2i-1} \geq 0$  for all  $i \geq 1$  such that  $2i \leq k'$ . Otherwise  $F_{s_1, \dots, s_{k'}} = 0$ . If  $k'$  is odd then  $F_{s_1, \dots, s_{k'}} = \left( \frac{x_i - y_i}{2t} \right)^{s_1} F_{s_2, \dots, s_{k'}}.$

The leading term of  $P_{i,k}$  in  $\frac{1}{t}$  is  $F_k = \left( \frac{x_i - y_i}{2t} \right)^k$ .  $\square$

*Proof of Proposition 8.* For a multi-index  $I = (i_1, \dots, i_n)$ ,

$$(66) \quad \frac{\partial K_t}{\partial x^I} = P_{1,i_1} \dots P_{n,i_n} K_t.$$

□

4.1.2. *Powers of  $t$  in  $w_\gamma(P, I)$ .* Let  $O(\gamma)$  be the sum of the orders of the local functionals  $I_{g(v), k(v)}$  for all  $v \in V(\gamma)$ . As a consequence of Corollary 8, if we group the terms in  $w_\gamma(P, I)$  by their powers of  $t$ , we see that

$$(67) \quad w_\gamma(P, I) = \int_{(\epsilon, L)^{|E(\gamma)|}} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} \sum_{-O(\gamma) \leq |J| \leq 0} t^{J-n/2} \Phi_J.$$

This formula requires some explanation. The outer integral is over the time variables. Secondly,

$$t^{J-n/2} = \prod_{e \in E(\gamma)} t_e^{j_e - n/2}.$$

In the exponential,  $Q_e = \|x_{v_1(e)} - x_{v_2(e)}\|^2$ .

The multi-index  $I : E(\gamma) \rightarrow \mathbb{Z}$  and for each  $I$ ,  $\Phi_I$  a sum of terms of the form

$$\prod_{v \in V(\gamma)} D_v \alpha(x_v)$$

where for each vertex  $v$ ,

$$(68) \quad D_v \alpha = D_{v,1} \alpha \dots D_{v,l} \alpha$$

is a product of differential operators applied to  $\alpha$ .

Then

$$(69) \quad f_\gamma[P, I](\mathbf{t}) = \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} \sum_{-O(\gamma) \leq |J| \leq 0} t^{J-n/2} \Phi_J$$

so that

$$(70) \quad w_\gamma(P, I) = \int_{(\epsilon, L)^{|E(\gamma)|}} f_\gamma[P, I].$$

where  $J$  is a multi-index.

4.1.3. *Spanning Tree Coordinates.* Choose a spanning tree  $T$  of  $\gamma$ . For each edge in the tree we define a coordinate  $y_e = x_{v_1(e)} - x_{v_2(e)}$ .

**Proposition 9.** *Given a spanning tree  $T$ , the coordinates  $y_e = x_{v_1(e)} - x_{v_2(e)}$  for  $e \in E(T)$  and*

$$(71) \quad w = x_1 + \dots + x_{|V(\gamma)|}$$

*form a coordinate system on  $\mathbb{R}^{n|V(\gamma)|}$ .*

*Proof.* This is a linear transformation from  $\mathbb{R}^{|V(\gamma)|}$  to  $\mathbb{R}^{|V(\gamma)|}$ . It is invertible if and only if it has trivial kernel. But if  $y_e = 0$  for all  $e \in T(\gamma)$  then  $x_i = x_j$  for all  $i$  and  $j$ . The condition that  $x_1 + \dots + x_{|V(\gamma)|} = 0$  then implies that  $x_i = 0$  for all  $i$ . □

The quadratic form  $Q(x) = \sum_{e \in E(\gamma)} Q_e(x)/4t_e$  can be written in the spanning tree coordinates as  $Q(w, y)$ . Let  $A$  be the matrix of  $Q(0, y)$ . Then  $A$  is an  $n(|V(\gamma)| - 1)$  by  $n(|V(\gamma)| - 1)$  matrix.

**Proposition 10.** *The quadratic form  $Q(w, y)$  is independent of  $w$ . The matrix  $B = (4 \prod_{e \in E(\gamma)} t_e) A$  has entries that are integer polynomials in  $\{t_e\}_{e \in E(\gamma)}$ . Consequently,  $P_\gamma = \det B$  is an integer polynomial in  $\{t_e\}_{e \in E(\gamma)}$ .*

*Proof.* For any edge  $e \in E(\gamma)$ , let  $f_1^e, \dots, f_{l(e)}^e$  be the unique path of edges in  $T$  connecting  $v_1(e)$  and  $v_2(e)$ . Then

$$x_{v_1(e)} - x_{v_2(e)} = \sum_{i=1}^{l(e)} (x_{v_1(f_i^e)} - x_{v_2(f_i^e)}) = \sum_{i=1}^{l(e)} y_{f_i^e}.$$

Therefore,

$$\begin{aligned} Q(x) &= \sum_{e \in E(\gamma)} Q_e(x)/4t_e \\ &= \sum_{e \in E(\gamma)} \left\| \sum_{i=1}^{l(e)} y_{f_i^e} \right\|^2 / 4t_e \\ &= Q(w, y) \end{aligned}$$

which clearly does not depend on  $w$ .

It is clear now that the matrix  $B$ , which is the matrix of the quadratic form  $(4 \prod_{e \in E(\gamma)} t_e) Q(0, y)$ , has entries which are polynomials in  $\{t_e\}_{e \in E(\gamma)}$  with integer coefficients.  $\square$

**Proposition 11.**

$$(72) \quad \det A = 4^{-n(|V(\gamma)|-1)} t^{-n(|V(\gamma)|-1)} P_\gamma$$

and

$$(73) \quad A^{-1} = \frac{1}{P_\gamma} C,$$

where  $C$  is a matrix with polynomial entries in  $t_e$ .

*Proof.* To prove the second statement, use Cramer's rule

$$(74) \quad B^{-1} = \frac{1}{\det B} \operatorname{adj}(B) = \frac{1}{P_\gamma} \operatorname{adj}(B)$$

and that  $A^{-1} = (4 \prod_{e \in E(\gamma)} t_e) B^{-1}$ . So, the statement follows by letting  $C = (4 \prod_{e \in E(\gamma)} t_e) \operatorname{adj}(B)$ .  $\square$

4.1.4. *Taylor Expansion of  $\Phi_I$ .* Now replace  $\Phi_J$  in  $f_\gamma[P, I](\mathbf{t})$  with its Taylor polynomial of degree  $N'$  in  $y$ ,  $\sum_{|K| \leq N'} c_{J,K} y^K$ , where  $N'$  is a non-negative integer to be determined. This gives

$$(75) \quad f_\gamma^{N'}[P, I](\mathbf{t}) = \sum_{\substack{|K| \leq N' \\ -O(\gamma) \leq |J| \leq 0}} \int_{\mathbb{R}^{n+|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} t^{J-n/2} c_{J,K} y^K dy dw$$

$$(76) \quad = \sum_{\substack{|K| \leq N', J \text{ even} \\ -O(\gamma) \leq |J| \leq 0}} \int_{\mathbb{R}^n} t^{J-n/2} c_{J,K} \mathcal{I}_A^K dw$$

where

$$(77) \quad \mathcal{I}_A^K(\mathbf{t}) = \int_{\mathbb{R}^{n(V(\gamma)-1)}} e^{-\langle y, Ay \rangle} y^K dy.$$

and  $c_{J,K}$  is a function of  $w$  only.

A change of variables gives

**Proposition 12.**

$$(78) \quad \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} |y^K| \leq C t_k^{\frac{1}{2}(|K| + (|V(\gamma)| - 1))}$$

for some constant  $C > 0$ .

And thus

**Proposition 13.**

$$(79) \quad |\mathcal{I}_A^K(\mathbf{t})| \leq C t_k^{\frac{1}{2}(|K| + n(|V(\gamma)| - 1))}$$

and

$$(80) \quad |f_\gamma^{N'}[P, I](\mathbf{t})| \leq C t_k^{\frac{n}{2}(|V(\gamma)| - 1) - R|E(\gamma)|\frac{n}{2} - RO(\gamma)}.$$

We can calculate  $\mathcal{I}_A^K$  rather explicitly:

**Theorem 6.**

$$(81) \quad \mathcal{I}_A^K(\mathbf{t}) = \frac{1}{P_\gamma^{(|K|+1)/2}} \mathcal{P}_A^K.$$

where  $\mathcal{P}_A^K$  is a homogeneous polynomial in  $\mathbf{t}$  of degree  $R(\gamma, n, K) = C_1(\gamma, n) + |K|C_2(\gamma, n)$  for constants  $C_1(\gamma, n)$  and  $C_2(\gamma, n)$  which are defined in the body of the proof of the theorem.

*Proof.* Writing  $y^K = y_{m_1} \dots y_{m_{|K|}}$ , we have

$$\begin{aligned} \mathcal{I}_A^K &= \int_{\mathbb{R}^{n(V(\gamma)-1)}} e^{-\langle y, Ay \rangle} y_{m_1} \dots y_{m_{|K|}} dy \\ &= \frac{(\sqrt{\pi})^{n(V(\gamma)-1)}}{\sqrt{\det A}} \sum_{\beta} \prod_{i=1}^{|K|/2} (A^{-1})_{m_{\beta_i^{(1)}}, m_{\beta_i^{(2)}}} \\ &= \frac{(\sqrt{\pi})^{n(V(\gamma)-1)}}{P_\gamma^{1/2}} 2^{n(|V(\gamma)|-1)} t^{n(|V(\gamma)|-1)/2} \frac{1}{P_\gamma^{|K|/2}} \sum_Q \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}}, \end{aligned}$$

where we have used Proposition 11 and Wick's Theorem on  $\mathbb{R}^n$ . Let

$$(82) \quad \mathcal{P}_A^K = (\sqrt{\pi})^{n(V(\gamma)-1)} 2^{n(|V(\gamma)|-1)} t^{n(|V(\gamma)|-1)/2} \sum_Q \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}}.$$

Recall the definition of  $C$  which is  $(4 \prod_{e \in E(\gamma)} t_e) \text{adj}(B)$ . But  $B$  is  $n(|V(\gamma)| - 1)$  by  $n(|V(\gamma)| - 1)$  and its entries are homogeneous of degree  $|E(\gamma)| - 1$  in  $\{t_e\}_{e \in E(\gamma)}$ . So  $\text{adj}(B)$  has entries of degree  $(|E(\gamma)| - 1)[n(|V(\gamma)| - 1) - 1]$ . Therefore  $C$  has entries of degree

$$(83) \quad (|E(\gamma)| - 1)[n(|V(\gamma)| - 1) - 1] + |E(\gamma)| = (|E(\gamma)| - 1)n(|V(\gamma)| - 1) + 1$$

This implies that  $\mathcal{P}_A^K$  is of homogeneous degree

$$(84) \quad R_\gamma(n, K) = n|E(\gamma)|(|V(\gamma)| - 1)/2 + \frac{|K|}{2}[ (|E(\gamma)| - 1)n(|V(\gamma) - 1) + 1]$$

With the definition of  $\mathcal{P}_A^K$  in hand, the theorem is now evident.  $\square$

4.1.5. *The structure of  $c_{J,K}$ .* In this section we prove that  $\Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} c_{J,K} dw$  is a local functional.

Recall that  $c_{J,K}(w) = \frac{\partial \Phi_J}{\partial y^K}(0, w)$  and that

$$\Phi_J = \prod_{v \in V(\gamma)} D_v \alpha(x_v)$$

where  $D_v$  is a product of differential operators. So

$$c_{J,K}(w) = \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(w).$$

$\tilde{D}_v$  is a product of differential operators. Finally, we see that

$$\Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(w) dw$$

is a local functional.

From (76) and (81) it is now clear that

**Corollary 4.**

$$(85) \quad f_\gamma^{N'}[P, I](\mathbf{t}) = \sum_{\substack{|K| \leq N', K \text{ even} \\ -O(\gamma) \leq |J| \leq 0}} \frac{\mathcal{P}_A^K(\mathbf{t})}{\mathcal{Q}_A^{J,K}(\mathbf{t})} \Psi_{J,K}(\alpha).$$

where  $\mathcal{Q}_A^{J,K}$  is of homogeneous degree

$$(86) \quad -|J| + \frac{n}{2}|E(\gamma)| + n(|V(\gamma)| - 1)(|E(\gamma)| - 1)(|K| + 1)/2.$$

Note that for a fixed spanning tree  $T$ ,  $x_v$  and  $\{y_e\}_{e \in E(T)}$  and  $w$  are related by a linear coordinate change. In order to calculate  $c_{J,K}$ , one would like to make this change explicit. Let  $e_1, \dots, e_l$  with  $v_1(e_1) = v$  and  $v_2(e_l) = w$  be the unique path in  $T$  connecting  $v$  and  $w$ . Then

$$x_v - x_w = \sum_{i=1}^l y_{e_i}$$

and thus we can express  $x_v$  in terms of  $\{y_e\}_{e \in E(T)}$  and  $w$  using the equation

$$x_v = \frac{1}{|V(\gamma)|} \left[ w + \sum_{w \neq v} (x_v - x_w) \right].$$

**4.2. Counterterms on  $\mathbb{R}^n$ : Error Bounds and Iteration.**

4.2.1. *Bounding the Error.* Let  $k = |E(\gamma)|$ . Assume that we order the edges so that  $t_1 \leq \dots \leq t_k$  and  $t_k^R \leq t_1$  so that  $\mathbf{t} \in A_R^k$  and that  $\mathbf{t} \in (0, 1)^k$ .

(87)

$$|f_\gamma[P, I](\mathbf{t}) - f_\gamma^{N'}[P, I](\mathbf{t})| \leq \sum_{|K|=N'+1} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} t_k^{-RO(\gamma)} t^{-n/2} d_K |y^K|$$

(88)

$$\leq \sum_{|K|=N'+1} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} t_k^{R(-O(\gamma)-|E(\gamma)|n/2)} d_K |y^K|$$

(89)

$$\leq \sum_{|K|=N'+1} \left( \int d_K dw \right) C_K t_k^{\frac{1}{2}(N'+1+n(|V(\gamma)|-1))} t_k^{R(-O(\gamma)-|E(\gamma)|n/2)}$$

using Proposition 12. In the formula above,

$$d_K(w) = \sum_J \sup_y \left| \frac{\partial \Phi_J}{\partial y^K}(y, w) \right|$$

But  $\Phi_J$  is a sum of terms of the form

$$f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}}$$

where  $f$  is a Schwartz function and  $I^h$  is a collection of multi-indices, one for each tail  $h$ , satisfying the condition  $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma)$ . This implies that  $\frac{\partial \Phi_J}{\partial y^K}$  is a sum of terms of the same form, but satisfying the condition  $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) + N' + 1$ .

But

$$\begin{aligned} \sup_y \left| f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| &\leq \sup_y |f| \cdot \sup_{y, w} \left| \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| \\ &\leq \sup_y |f| \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \end{aligned}$$

where  $p_h = |I^h|$ . Note that  $\sup_y |f|$  is a Schwartz function in the variable  $w$ . Thus,

$$\int d_K dw \leq \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h}$$

where the summation is over multi-indices  $p : T(\gamma) \rightarrow \mathbb{Z}^{\geq 0}$  such that  $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$ .

In conclusion, we have shown that

**Theorem 7.**

(90)

$$|f_\gamma[P, I](\mathbf{t}) - f_\gamma^{N'}[P, I](\mathbf{t})| \leq \left( \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{\frac{1}{2}(N'+1)+(|V(\gamma)|-1)\frac{n}{2}-R(O(\gamma)+|E(\gamma)|\frac{n}{2})}$$

where the summation is over multi-indices  $p : T(\gamma) \rightarrow \mathbb{Z}^{\geq 0}$  such that  $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$ .



4.2.2. *Inductive Construction of the Counterterms.* We shall use Corollary 3, which gives a finite cover of  $(0, 1)^k$  given by sets of the form  $\overline{E}_R^{I(1)} \cap \overline{E}_R^{I(2)} \cdots \cap \overline{E}_R^{I(p)}$ , for  $p \leq k$ . The sets  $E_R^I$  are defined in Definition 3.

Also we shall need Theorem 5 which states that for  $E_R^I$  where  $I$  is a sequence of the form  $1 < i_1 < \cdots < i_m \leq k$  with  $m < k$ , we have  $E_R^I \subseteq A_{R^{2^m}}^{i_m}$ , where

$$(91) \quad A_{R^{2^m}}^{i_m} = \{t_1 < t_2 < \cdots < t_k : t_{i_m} < t_{i_m+1}^{R^{2^m}} \text{ and } t_{i_m}^{R^{2^m}} < t_1\}.$$

**Theorem 8.** *For any sequence  $I^{(1)}, \dots, I^{(p)}$  as in Corollary 3, for nonnegative integers  $N'_1, \dots, N'_p$ .*

$$(92) \quad |f_\gamma[P, I](\mathbf{t}) - f_\gamma^{N'_1, \dots, N'_p}[P, I](\mathbf{t})| \leq \|\alpha\|_{O(\gamma)}^{|T(\gamma)|} \sum_i C_i t^{d_i}$$

where  $d_i = d_i(N'_1, \dots, N'_i)$  increases linearly in  $N'_i$  for  $N'_1, \dots, N'_{i-1}$  fixed and sufficiently large, and where  $f_\gamma^{N'_1, \dots, N'_p}[P, I](\mathbf{t})$  is defined by iterative Taylor expansion of  $f_\gamma[P, I](\mathbf{t})$  as illustrated in the proof of the theorem.

*Proof.* Fix an ordering of the edge set so that we can identify  $E(\gamma) = \{e_1, \dots, e_k\} = \{1, \dots, k\}$ . The procedure should be carried out in

$$(93) \quad S_\sigma = \{\mathbf{t} \in (0, \infty)^k : t_{\sigma(1)} < \cdots < t_{\sigma(k)}\}.$$

for each permutation  $\sigma \in S_k$ . However, we shall work in  $S_{\text{id}}$  for notational simplicity.

For general  $p$ , we consider the cover of  $S_{\text{id}}$  by sets of the form  $\overline{E}_R^{I(1)} \cap \overline{E}_R^{I(2)} \cdots \cap \overline{E}_R^{I(p)}$ , for each of the  $k!$  possible orderings of the edges of  $\gamma$ .

For illustrative purposes and notational simplicity we prove the main theorem only for  $p = 2$ . The inductive step in the proof of the general case is similar. Let  $i^{(1)} = i_{m(1)}^{(1)}$  and let  $R_1 = R^{s_{m(1)+1}}$  and  $R_2 = R^{s_{m(2)+1}}$  so that we are working within

$$\begin{aligned} \overline{E}_R^{I(1)} \cap \overline{E}_R^{I(2)} &\subseteq \overline{A}_{R_1}^{i^{(1)}} \cap \overline{A}_{R_2}^k \\ &= \{\mathbf{t} \in S_{\text{id}} : t_{i^{(1)}}^{R_1} \leq t_1 \text{ and } t_{i^{(1)}} \leq t_{i^{(1)}+1}^{R_1} \text{ and } t_k \leq t_{i^{(1)}+1}^{R_2}\} \end{aligned}$$

where the inclusion follows from Theorem 5. Note that  $i_{m(2)}^{(2)} = k$ .

The collection of edges  $e_1, \dots, e_{i^{(1)}}$  determines a subgraph of  $\gamma$ , which we denote by  $\gamma'$ . The remaining edges  $e_{i^{(1)}+1}, \dots, e_k$  form the edge set of  $\gamma/\gamma'$ .

A tail  $h \in T(\gamma')$  will either be in  $T(\gamma)$  or will be one of the two half edges forming an edge in  $E(\gamma)$ . In the formula above,  $T(\gamma', \gamma) = T(\gamma') \cap T(\gamma)$  and  $E(\gamma', \gamma) = E(\gamma') \cup F(\gamma', \gamma)$ , where  $F(\gamma', \gamma)$  is the set of all edges in  $\gamma$  for which one half edge making up the edge is a tail in  $\gamma'$ . Let  $V(\gamma', \gamma)$  denote the vertices not in  $\gamma'$  that are incident on an edge in  $F(\gamma', \gamma)$ .

The integral in the formula for  $f_\gamma[P, I]$  is over  $\mathbb{R}^{n|V(\gamma)|}$  and we can order the integration so that we integrate first with respect to the vertices in  $V(\gamma')$ . This inner integral, which is of the form

$$(94) \quad \int_{\mathbb{R}^{n|V(\gamma')|}} \prod_{v \in V(\gamma')} a^{I^{v^1}, \dots, I^{v^k}}(x_v) \prod_{e \in E(\gamma', \gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h \in T(\gamma', \gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}}$$

Let us use  $f_{\gamma',\gamma}[P, I]$  to denote the integral above. It is a function of  $\{t_e\}_{e \in E(\gamma',\gamma)}$  and  $x_v$  for  $v \in V(\gamma',\gamma)$ . Let  $f_{\gamma/\gamma'}[P, I, I']$  be as in (97) but with the local functional  $I'$  used for the distinguished vertex in  $\gamma/\gamma'$ .

Use the same procedure which led to Corollary 4 and Theorem 7. We have that  $|f_{\gamma',\gamma}[P, I] - f_{\gamma',\gamma}^{N'_1}[P, I]|$  is less than or equal to

$$\left( \sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_{h(e)}} \right) t_{i(1)}^{\frac{1}{2}N'_1 + C(\gamma',n,R_1)}$$

where

$$C(\gamma',n,R_1) = \frac{1}{2} + (|V(\gamma')| - 1)\frac{n}{2} - R_1 \left( O(\gamma') - |E(\gamma')|\frac{n}{2} \right).$$

But

$$\|K_{t_e}\|_{p_{h(e)}} \leq C t_e^{-\frac{n}{2} - p_h(e)} \leq C t_{i(1)+1}^{-\frac{n}{2} - p_h(e)}$$

for some constant  $C$ , and thus

$$\begin{aligned} \prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_{h(e)}} &\leq C t_{i(1)+1}^{-|F(\gamma',\gamma)|\frac{n}{2} - \sum_{e \in F(\gamma',\gamma)} p_{h(e)}} \\ &\leq C t_{i(1)+1}^{-|E(\gamma')|\frac{n}{2} - O(\gamma') - N_1} \end{aligned}$$

for some constant  $C$ .

Thus,  $|f_{\gamma',\gamma}[P, I] - f_{\gamma',\gamma}^{N'_1}[P, I]|$  is less than or equal to

$$\left( \sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \right) t_{i(1)+1}^{\frac{R_1}{2}N'_1 + R_1 C(\gamma',n,R_1) - |E(\gamma')|\frac{n}{2} - O(\gamma') - N'_1}$$

So as long as  $R_1 > 2$ ,  $|f_{\gamma',\gamma}[P, I] - f_{\gamma',\gamma}^{N'_1}[P, I]|$  and consequently

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma',\gamma}[P, I]] - f_{\gamma/\gamma'}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]]|$$

will be bounded by a power of  $t$  which grows linearly with  $N'_1$ .

To finish the argument, we need to show the same thing for

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]] - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]]|.$$

From Proposition 13,

$$|f_{\gamma',\gamma}^{N'_1}[P, I]| \leq C t_{i(1)}^{\frac{n}{2}(|V(\gamma')|-1) - R_1|E(\gamma')|\frac{n}{2} - R_1 O(\gamma')}.$$

where we consider the  $t_e$  variables in  $f_{\gamma',\gamma}^{N'_1}[P, I]$  for  $e \in F(\gamma',\gamma)$  to be fixed. Let  $C_1(\gamma',\gamma,n,R_1)$  be the power of  $t_{i(1)}$  in the inequality above.

We are able to bound  $|f_{\gamma/\gamma'}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]] - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]]|$  by

$$C \|\alpha\|_l^{|T(\gamma')|} t_k^{\frac{1}{2}N'_2 + \frac{1}{2} + (|V(\gamma/\gamma')|-1)\frac{n}{2} - R_1(O(\gamma) + N'_1) + R_1 C_1(\gamma',\gamma,n,R_1)}$$

In conclusion, using the triangle inequality and that we can bound

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma',\gamma}[P, I]] - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma',\gamma}^{N'_1}[P, I]]| \leq C_1 t_k^{d_1(N'_1)} + C_2 t_k^{d_2(N'_1, N'_2)}$$

where by  $d_1(N_1)$  grows linearly in  $N'_1$  and  $d_2(N_1, N_2)$  grows linearly in  $N_2$  for  $N_1$  fixed.  $\square$

**4.3. Counterterms on the Euclidean Upper Half Space.** The Dirichlet heat kernel on the upper half space  $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$  with the Euclidean metric is given by

$$(95) \quad K_t(x, y) = (4\pi t)^{-n/2} [e^{-|x-y|^2/4t} - e^{-|x-y^*|^2/4t}],$$

where  $y^*$  is the reflection through the hyperplane  $y_n = 0$ .

Note that  $K_t$  solves the heat equation, for  $y \in \partial \mathbb{H}_+^n$ ,  $K_t(x, y) = 0$ , and

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{H}^n} K_t(x, y) \phi(y) dy = \phi(x)$$

for any  $\phi \in C^\infty(\mathbb{H}^n)$ .

Similarly to 4.1, we form

$$(96) \quad w_\gamma(P, I) = \int_{(\epsilon, L)^{|E(\gamma)|}} f_\gamma[P, I].$$

but now

$$(97) \quad f_\gamma[P, I](\mathbf{t}) = \sum_{\beta} \int_{\mathbb{H}^n |V(\gamma)|} e^{-\sum_{e \in E(\gamma)} Q_e^{(\beta_e)}/4t_e} \sum_{-O(\gamma) \leq |J| \leq 0} t^{J-n/2} \Phi_{J, \beta}$$

where  $Q_e^{(1)} = \|x_{v_1(e)} - x_{v_2(e)}\|^2$  and  $Q_e^{(-1)} = \|x_{v_1(e)} - x_{v_2(e)}^*\|^2$  and  $\beta$  ranges over all functions  $E(\gamma) \rightarrow \{-1, 1\}$ . As in 4.1, we wish to apply Wick's theorem after taking the Taylor expansion of  $\Phi_{J, \beta}$ .

**4.3.1. Coordinate System on  $\mathbb{H}^n |V(\gamma)|$ .** Using the decomposition  $\mathbb{H}^n = \mathbb{R}^{(n-1)} \times \mathbb{R}^{\geq 0}$  introduce the coordinates  $x_v = (\bar{x}_v, x_{v,n})$ . Split the integral into an integral over  $\mathbb{R}^{(n-1)|V(\gamma)|}$  followed by an integral over  $(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$ . The quadratic form with these coordinates becomes

$$\sum_{e \in E(\gamma)} \|\bar{x}_{v_1(e)} - \bar{x}_{v_2(e)}\|^2 / 4t_e$$

plus the part depending on the variables  $x_{v,n}$

$$\sum_{e \in \beta^{-1}(1)} |x_{v_1(e),n} - x_{v_2(e),n}|^2 / 4t_e + \sum_{e \in \beta^{-1}(-1)} |x_{v_1(e),n} + x_{v_2(e),n}|^2 / 4t_e$$

We shall only concentrate on the part of the quadratic form depending on the variables  $x_{v,n}$  since the integral over the variables  $\bar{x}_v$  can be treated by the methods of 4.1.

Choose an ordering on the set of vertices and consider the basis,  $f_{|V(\gamma)|} = e_1 + \dots + e_n$ ,  $f_1 = e_1 - e_2, \dots, f_{|V(\gamma)|-1} = e_{|V(\gamma)|-1} - e_{|V(\gamma)|}$ . This induces the coordinate system  $u, z_1, \dots, z_{|V(\gamma)|-1}$  on  $\mathbb{R}^{|V(\gamma)|}$  related to standard coordinates by

$$\begin{aligned} x_{1,n} &= u + z_1 = u + \tilde{z}_1 \\ x_{2,n} &= u + z_2 - z_1 = u + \tilde{z}_2 \\ &\dots \\ x_{|V(\gamma)|-1,n} &= u + z_{|V(\gamma)|-1} - z_{|V(\gamma)|-2} = u + \tilde{z}_{|V(\gamma)|-1} \\ x_{|V(\gamma)|,n} &= u - z_{|V(\gamma)|-1} = u + \tilde{z}_{|V(\gamma)|} \end{aligned}$$

In this coordinate system the second part of the quadratic form becomes

$$\sum_{e \in \beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2 / 4t_e + \sum_{e \in \beta^{-1}(-1)} |2u + \tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2 / 4t_e.$$

Let  $P$  be the plane spanned by  $f'_i$  for  $i$  between 1 and  $|V(\gamma)| - 1$  the subscript indicates the dependence on  $\{t_e\}_{e \in E(\gamma)}$ . Then for  $u \geq 0$

$$u(e_1 + \cdots + e_{|V(\gamma)|}) + P$$

intersects  $(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$  in a bounded set (in particular a simplex) whose projection onto  $P$  we denote  $P_u$ .

**4.3.2. Taylor Expansion of  $\Phi_J$ .** For a fixed spanning tree of  $\gamma$ , choose spanning tree coordinates on  $\mathbb{R}^{(n-1)|V(\gamma)|}$ ,  $\bar{y}_e = \bar{x}_{v_1(e)} - \bar{x}_{v_2(e)}$  and  $\bar{w} = \bar{x}_1 + \cdots + \bar{x}_{|V(\gamma)|}$  on  $\mathbb{R}^{(n-1)|V(\gamma)|}$ . As in the previous section, choose coordinates  $z_1, \dots, z_{|V(\gamma)|-1}$  and  $u$  on  $(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$ .

In these coordinates, the quadratic form  $\sum_{e \in E(\gamma)} Q_e / t_e$  decomposes into a sum of three terms

$$Q(\bar{y}, \mathbf{t}) + Q^{(\beta)}(z, u, \mathbf{t}) + Q^{(\beta)}(u, \mathbf{t}),$$

where as in the proof of Proposition 10,

$$Q(\bar{y}, \mathbf{t}) = \sum_{e \in E(\gamma)} \left\| \sum_{i=1}^{l(e)} \bar{y}_{f_i^e} \right\|^2 / 4t_e,$$

$$\begin{aligned} Q^{(\beta)}(z, u, \mathbf{t}) &= \sum_{e \in \beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2 / 4t_e + \sum_{e \in \beta^{-1}(-1)} |\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2 / 4t_e \\ &\quad + \sum_{e \in \beta^{-1}(-1)} u(\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}) / t_e \end{aligned}$$

and

$$Q^{(\beta)}(u, \mathbf{t}) = \left( \sum_{e \in \beta^{-1}(-1)} t_e^{-1} \right) u^2.$$

We shall Taylor expand  $\Phi_J$  in  $f_\gamma[P, I]$  in both  $\bar{y}$  and  $z$  to order  $N'$ . For  $f_\gamma^{N'}[P, I]$  we have a sum of integrals of the form

$$\int_{\mathbb{R}^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-Q(\bar{y}, \mathbf{t}) - Q^{(\beta)}(z, u, \mathbf{t}) - Q^{(\beta)}(u, \mathbf{t})} t^{J-n/2} c_{J, K, K', \beta} \bar{y}^K z^{K'} d\bar{y} d\bar{w} dz du,$$

over  $|K| + |K'| \leq N'$ ,  $J$  even,  $-O(\gamma) \leq |J| \leq 0$  and  $\beta$  functions  $E(\gamma) \rightarrow \{-1, 1\}$ . Also,  $c_{J, K, K', \beta}$ , the Taylor coefficient is a function of  $w$  and  $u$  only.

The integral over  $\bar{y}$  gives an answer like that of Theorem 6 and Corollary 4, but with the dimension  $n$  replaced by  $n - 1$  in all the formulas. The integral over  $z$  exists because  $P_u$  is a bounded set. Let

$$(98) \quad \phi_{K'}(u, \mathbf{t}) = \int_{P_u} e^{-Q^{(\beta)}(z, u, \mathbf{t})} z^{K'} dz$$

Let

$$(99) \quad \mathcal{I}_{K,K'}(u) = \int_{P_u} \int_{\mathbb{R}^{(n-1)(|V(\gamma)|-1)}} e^{-Q(\bar{y}, \mathbf{t}) - Q^{(\beta)}(z, u, \mathbf{t})} \bar{y}^K z^{K'} d\bar{y} dz$$

$$(100) \quad = \frac{\mathcal{P}_A^K}{\mathcal{Q}_A^K} \phi_{K'}(u, \mathbf{t})$$

where  $\mathcal{P}_A^K(\mathbf{t})$  and  $\mathcal{Q}_A^K(\mathbf{t})$  are defined analogously to the functions in 4.

Then  $f_\gamma^{N'}[P, I]$  becomes a sum of integrals

$$(101) \quad t^{J-n/2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R} \geq 0} e^{-Q^{(\beta)}(u, \mathbf{t})} \mathcal{I}_{K,K'}(u) c_{J,K,K',\beta}(\bar{w}, u) du d\bar{w}$$

$$(102) \quad = t^{J-n/2} \frac{\mathcal{P}_A^K}{\mathcal{Q}_A^K} \int_{\mathbb{H}} \psi_{K'}(u, \mathbf{t}) c_{J,K,K',\beta}(\bar{w}, u) du d\bar{w}$$

$$(103) \quad = \frac{\mathcal{P}_A^K}{\mathcal{Q}_A^{J,K}} \Psi_{J,K,K'}(\mathbf{t}, \alpha)$$

where  $\psi_{K'}(u, \mathbf{t}) = e^{-Q^{(P)}(u, \mathbf{t})} \phi_{K'}(u, \mathbf{t})$ . This is not quite the product of a function of  $\mathbf{t}$  and a local functional because  $\psi_{K'}(u, \mathbf{t})$  depends on  $\mathbf{t}$ .

Recall that  $c_{J,K,K'}(\bar{w}, u) = \frac{\partial \Phi_J}{\partial \bar{y}^K \partial z^{K'}}(0, \bar{w}, 0, u)$  and that

$$\Phi_J = \prod_{v \in V(\gamma)} D_v \alpha(x_v)$$

where  $D_v$  is a product of differential operators. So

$$c_{J,K,K'}(\bar{w}, u) = \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(\bar{w}, u).$$

$\tilde{D}_v$  is a product of differential operators. Finally, we see that the integrand in

$$(104) \quad \Psi_{J,K,K'}(\mathbf{t}, \alpha) = \int_{\mathbb{H}^n} \psi_{K'}(u, \mathbf{t}) \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(\bar{w}, u) du d\bar{w}$$

is almost a local functional  $\psi_{K'}(u, \mathbf{u})$  is the  $\mathbf{t}$ -dependent factor. For  $t_1 \leq \dots \leq t_k$ , We do have control on the  $\mathbf{t}$  dependence

$$(105) \quad |\psi_{K'}(\sqrt{t_k} u, \mathbf{t})| \leq C t_k^{\frac{1}{2}(|K'| + |V(\gamma)| - 1)}.$$

We will show how to the renormalization procedure can be adapted to this situation in the next section.

4.3.3. *Bounding the Error.* Note that

**Proposition 14.**

$$(106) \quad \int_{P_{\sqrt{t_k} u}} \int_{\mathbb{R}^{(n-1)(|V(\gamma)|-1)}} e^{-Q(\bar{y}, \mathbf{t}) - Q^{(\beta)}(z, \sqrt{t_k} u, \mathbf{t})} |\bar{y}^K| |z^{K'}| d\bar{y} dz \leq t_k^{\frac{1}{2}(N' + n(|V(\gamma)| - 1))}$$

where  $|K| + |K'| = N'$ .

Assume that we have ordered the set of edges so that  $t_1 \leq \dots \leq t_k$  and  $t_k^R \leq t_1$  for some  $R > 1$ . By Taylor's theorem  $|f_\gamma[P, I](\mathbf{t}) - f_\gamma^{N'}[P, I](\mathbf{t})|$  is bounded above by a sum of terms of the form

$$(107) \quad \int_{\mathbb{R}_u^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} t_k^{-RO(\gamma)} t^{-n/2} d_{K, K', \beta} |\bar{y}^K| |z^{K'}|$$

over multi-indices  $|K| + |K'| = N' + 1$  and  $\beta : E(\gamma) \rightarrow \{-1, 1\}$ . Each such term is bounded above by

$$(108) \quad \leq \int_{\mathbb{R}_u^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} t_k^{R(-O(\gamma) - |E(\gamma)|n/2)} d_{K, K', \beta} |\bar{y}^K| |z^{K'}|$$

$$(109) \quad \leq \left( \int d_{K, K', \beta}(\bar{w}, \sqrt{t_k}u) C_{K, K'}(u) d\bar{w} du \right) t_k^{\frac{1}{2} + \frac{1}{2}(N' + 1 + n(|V(\gamma)| - 1))} t_k^{R(-O(\gamma) - |E(\gamma)|n/2)}$$

$$(110) \quad \leq \left( \int e_{K, K', \beta}(w) C_{K, K'}(u) d\bar{w} du \right) t_k^{\frac{1}{2}(N' + 2 + n(|V(\gamma)| - 1))} t_k^{R(-O(\gamma) - |E(\gamma)|n/2)}$$

where

$$(111) \quad C_{K, K'}(u) = e^{-|\beta^{-1}(-1)|u^2} \int_{P_u} |z|^{K'} dz \int_{\mathbb{R}^{(n-1)(|V(\gamma)| - 1)}} e^{-\bar{Q}(\bar{y})} \bar{y}^K d\bar{y}$$

is a Schwartz function in  $u$  for  $\beta^{-1}(-1) \neq 0$ , and

$$e_{K, K', \beta}(w) = \sup_u d_{K, K', \beta}(\bar{w}, u).$$

It remains to understand the integral

$$\int e_{K, K', \beta}(\bar{w}) C_{K, K'}(u) d\bar{w} du$$

in terms of the field  $\alpha$  and its derivatives. In the formula above,

$$d_{K, K', \beta}(\bar{w}) = \sum_J \sup_{\bar{y}, z} \left| \frac{\partial \Phi_J}{\partial \bar{y}^K z^{K'}} \right|$$

so

$$e_{K, K', \beta}(\bar{w}) = \sum_J \sup_{\bar{y}, z, u} \left| \frac{\partial \Phi_J}{\partial \bar{y}^K z^{K'}} \right|.$$

But  $\Phi_J$  is a sum of terms of the form

$$f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}}$$

where  $f$  is a Schwartz function and  $I^h$  is a collection of multi-indices, one for each tail  $h$  satisfying the condition  $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma)$ . This implies that  $\frac{\partial \Phi_J}{\partial \bar{y}^K z^{K'}}$  is a sum of terms of the same form, but satisfying the condition  $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) + N' + 1$ .

So

$$\begin{aligned} \sup_{\bar{y}, z, u} \left| f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| &\leq \sup_{\bar{y}, z, u} |f| \cdot \sup_{\bar{y}, \bar{w}, z, u} \left| \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| \\ &\leq \sup_{\bar{y}, z, u} |f| \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \end{aligned}$$

where  $p_h = |I^h|$ . Note that  $\sup_{\bar{y}, z, u} |f|$  is a Schwartz function in the variable  $\bar{w}$ . Thus,

$$\int e_{K, K', \beta}(w) C_{K, K'}(u) d\bar{w} du \leq \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h}$$

where the summation is over multi-indices  $p : T(\gamma) \rightarrow \mathbb{Z}^{\geq 0}$  such that  $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$ .

In conclusion, we have shown that

**Theorem 9.**

(112)

$$|f_{\gamma, P, I}(t_e) - f_{\gamma, P, I}^{N'}(t_e)| \leq \left( \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{\frac{1}{2}(N'+2) + (|V(\gamma)|-1)\frac{\alpha}{2} - R(O(\gamma)-|E(\gamma)|\frac{\alpha}{2})}$$

where the summation is over multi-indices  $p : T(\gamma) \rightarrow \mathbb{Z}^{\geq 0}$  such that  $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$ .

**4.3.4. Inductive Construction of the Counterterms.** We will only note the differences from Section 4.2.2.

The proof of Theorem 8, which was given only in the case  $p = 2$  involves two steps. In the first step, we show that  $\left| f_{\gamma', \gamma}[P, I](\mathbf{t}) - f_{\gamma', \gamma}^{N'_1}[P, I](\mathbf{t}) \right|$  is bounded by  $t_k$  to a power that grows linearly in  $N'_1$ . This implies that

$$\left| f_{\gamma}[P, I](\mathbf{t}) - f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t}) \right|$$

is bounded by a power of  $t_k$  that grows linearly in  $N'_1$ , where we have used that

$$f_{\gamma}[P, I] = f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}[P, I]].$$

In the second step, we show that

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t}) - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})|$$

is bounded by a power of  $t_k$  that grows linearly in  $N'_2$  for  $N'_1$  fixed. This will require a slight modification from the procedure of Section 4.2.2. To construct  $f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$ , we start with  $f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$  which is formed from the Feynman rules applied to the pointed graph  $\gamma/\gamma'$ , where we place the functional  $f_{\gamma', \gamma}^{N'_1}[P, I]$  on the distinguished vertex.

In the case of  $\mathbb{H}^n$ ,  $f_{\gamma', \gamma}^{N'_1}[P, I]$  is no longer a sum of functions of  $t_1, \dots, t_{i(1)}$  multiplied by local functionals applied to the inputs on the tails of  $\gamma'$ . Now the local functional depends on  $\mathbf{t}$  in the integrand.

When forming  $f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$  from  $f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$  only take the Taylor expansion of the factors in the integrand of  $f_{\gamma', \gamma}^{N'_1}[P, I]$  that do not depend on  $\mathbf{t}$ . That is, in the integrand of each  $\Psi_{J, K, K'}$ , neglect the first factor  $\psi_{K'}(u, \mathbf{t})$  and only take the Taylor expansion of the second factor. Because  $|\psi_{K'}(\sqrt{t_k}u, \mathbf{t})| \leq t_{i(1)}^{\frac{1}{2}(|V(\gamma)|-1)}$ , this will contribute factor of  $t_k^{\frac{R_1}{2}(|V(\gamma)|-1)}$  to the bound on

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t}) - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})|.$$

so the overall power of  $t_k$  will in fact be the same as in the case of  $\mathbb{R}^n$ .

**4.4. Counterterms on a Compact Manifold.** The asymptotic formula for the scalar heat kernel  $K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y) t^i$  states precisely that there exists some sequence of smooth functions  $\phi_i$  on  $M \times M$  such that

$$(113) \quad \|K_t(x, y) - \Psi(x, y) e^{-d(x, y)^2/4t} \sum_{i=0}^N \phi_i(x, y) t^i\|_l = O(t^{N-n/2-l}),$$

where  $\Psi$  is smooth function supported on a neighborhood of the diagonal in  $M \times M$ . Let

$$K_t^N(x, y) = \Psi(x, y) e^{-d(x, y)^2/4t} \sum_{i=0}^N \phi_i(x, y) t^i.$$

Beginning from (63), for each edge, we replace  $K_t$  with  $K_t^N$ . We let  $f_\gamma^N[P, I]$  denote the result of making all  $|E(\gamma)|$  of such substitutions. Assume that we've ordered the edges so that  $t_1 \leq \dots \leq t_k$ . Then

$$|f_\gamma[P, I](\mathbf{t}) - f_\gamma^N[P, I](\mathbf{t})| \leq \|\alpha\|_{O(\gamma)}^{|T(\gamma)|} \sum_{i=1}^k C_i t_i^{N-n/2-p_i} \prod_{j \neq i} t_j^{-p_j-n/2}$$

for some nonnegative integers  $p_j$  with  $\sum_j p_j \leq O(\gamma)$  and some constants  $C_i$ . Thus

$$|f_\gamma[P, I](\mathbf{t}) - f_\gamma^N[P, I](\mathbf{t})| \leq C \|\alpha\|_{O(\gamma)}^{|T(\gamma)|} t_k^{N-O(\gamma)-|E(\gamma)|\frac{n}{2}}.$$

An analogous statement to Proposition 8 can be made for  $K_t^N$  giving that

**Proposition 15.** *For the heat kernel in (64)*

$$(114) \quad \frac{\partial K_t^N}{\partial x_i^k} = P_{i,k} e^{-d(x, y)^2/4t}$$

where  $P_{i,k}$  is a Laurent polynomial in  $t$  of degree between  $-k$  and  $N$ .

Therefore, we have a formula

$$(115) \quad f_\gamma^N[P, I](\mathbf{t}) = \int_{U^{|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} \sum_{-O(\gamma) \leq |J| \leq |E(\gamma)|N} t^{J-n/2} \Phi_J$$

where  $Q_e = d^2(x_{v_1(e)}, x_{v_2(e)})$ .



4.4.1. *Spanning Tree Coordinates.* Choose a spanning tree  $T$  and vertex  $v_0$  of  $\gamma$ . For any vertex  $v$  of  $\gamma$ , there is a unique path  $e_1^v, \dots, e_{l(v)}^v$  from  $v_0$  to  $v$ . We can then make the inductive definition

**Definition 5.** *Spanning tree coordinates are defined by  $w = x_{v_0}$  and  $\{y_e\}_{e \in T}$  where  $y_e$  is defined inductively so that  $x_{v_2(e)} = \exp_{x_{v_1(e)}(y_{e_1}, \dots, y_{e_l})}(y_e)$  where  $e_1, \dots, e_l$  is the unique path from  $v_0$  to  $v_1(e)$ .*

The reason for introducing these coordinates is that for all  $e \in T$ ,

$$d(x_{v_1(e)}, x_{v_2(e)}) = \|y_e\|.$$

More explicitly, the spanning tree is the union of  $q$  maximal paths originating at  $v_0$ . Let us denote the  $i$ -th such path by  $e_1^{(i)}, \dots, e_{l_i}^{(i)}$ . Let  $V_1^{(i)}$  be a neighborhood of the zero section in  $TM$  such that  $\Phi_1^{(i)} : V_1^{(i)} \rightarrow U_1^{(i)} \subseteq M \times M$  given by  $(x_{v_0}, y_{e_1^{(i)}}) \mapsto (x_{v_0}, \exp_{x_{v_0}}(y_{e_1^{(i)}}))$  is a diffeomorphism. Inductively, given  $\Phi_{j-1}^{(i)} : V_{j-1}^{(i)} \rightarrow U_{j-1}^{(i)} \subseteq M^j$  let  $\exp_{j-1}^{(i)} = p_j \circ \Phi_{j-1}^{(i)}$ , where  $p_j$  is the projection onto the last factor. Now let  $V_j^{(i)}$  be a neighborhood of the zero section in  $(\exp_{j-1}^{(i)})^* TM$  such that the map  $\Phi_j^{(i)} : V_j^{(i)} \rightarrow U_j^{(i)} \subseteq M^{j+1}$  given by

$$(x_{v_0}, y_{e_1^{(i)}}, \dots, y_{e_j^{(i)}}) \mapsto \left( \Phi_{j-1}^{(i)}(x_{v_0}, y_{e_1^{(i)}}, \dots, y_{e_{j-1}^{(i)}}), \exp_{\exp_{j-1}^{(i)}(x_{v_0}, y_{e_1^{(i)}}, \dots, y_{e_{j-1}^{(i)}})}(y_{e_j^{(i)}}) \right)$$

is a diffeomorphism.

We then take the fiber product over  $M$  of the maps  $\Phi_{l_i}^{(i)}$  which produces the desired diffeomorphism

$$V_{l_1}^{(1)} \times_M \dots \times_M V_{l_m}^{(q)} \rightarrow M^{|V(\gamma)|}.$$

4.4.2. *Taylor Expanding  $\Phi_J$  and Bounding the Error.* Taking the Taylor expansion of  $\Phi$  with respect to  $\{y_e\}_{e \in E(\gamma)}$ .

(116)

$$f_\gamma^{N, N'}[P, I](\mathbf{t}) = \sum_{\substack{|K| \leq N' \\ -O(\gamma) \leq |J| \leq N|E(\gamma)|}} \int_{U^{|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t} t^{J-n/2} c_{J,K}(w) y^K dy dw$$

(117)

$$= \sum_{\substack{|K| \leq N', K \text{ even} \\ -O(\gamma) \leq |J| \leq N|E(\gamma)|}} \int_U t^{J-n/2} c_{J,K}(w) \mathcal{I}_A^K(w, \mathbf{t}) dw$$

(118)

$$= \sum_{\substack{|K| \leq N', K \text{ even} \\ -O(\gamma) \leq |J| \leq N|E(\gamma)|}} t^{J-n/2} \Psi_K(\mathbf{t}, \alpha) dw$$

This differs from the case of  $\mathbb{R}^n$  where  $\mathcal{I}_A^K(w, \mathbf{t})$  does not depend on  $w$ . Note that  $\int_U c_{J,K}(w) \mathcal{I}_A^K(w, \mathbf{t}) dw$  is not a local functional due to the factor of  $\mathcal{I}_A^K(w, \mathbf{t})$ , which depends on  $\mathbf{t}$ . We will say below why the procedure to construct the counterterms still works.

We have the bound

**Proposition 16.**

$$(119) \quad \int_{U^{|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} |y|^K dy \leq C t_k^{\frac{1}{2}|K| + \frac{N}{2}(|V(\gamma)|-1)}$$

*Proof.* This follows from the fact that

$$e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} \leq e^{-\sum_{e \in E(T)} Q_e/4t_e} = e^{-\sum_{e \in E(T)} \|y_e\|^2/4t_e}$$

□

Using Proposition 16, the bound on

$$|f_\gamma^N[P, I](\mathbf{t}) - f_\gamma^{N, N'}[P, I](\mathbf{t})|$$

can be established as in the proof of Theorem 7.

As in the case of  $\mathbb{H}^n$ , in the proof of Theorem 8 we make the following modification. When forming  $f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$  from  $f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})$  only take the Taylor expansion of the factors in the integrand of  $f_{\gamma', \gamma}^{N'_1}[P, I]$  that do not depend on  $\mathbf{t}$ . That is, in the integrand of each term  $\Psi_{J, K}$ , neglect the factor  $\mathcal{I}_A^K(w, \mathbf{t})$  and only take the Taylor expansion of the other factor.

Because  $|\mathcal{I}_A^K(w, \mathbf{t})| \leq t_{i(1)}^{\frac{N}{2}(|V(\gamma)|-1)}$ , this will contribute factor of  $t_k^{\frac{N}{2}(|V(\gamma)|-1)}$  to the bound on

$$|f_{\gamma/\gamma'}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t}) - f_{\gamma/\gamma'}^{N'_2}[P, I, f_{\gamma', \gamma}^{N'_1}[P, I]](\mathbf{t})|.$$

so the overall power of  $t_k$  will be the same as in the case of  $\mathbb{R}^n$ .

**4.5. Counterterms on a Compact Manifold with Boundary.** The renormalization procedure can also be carried out in the case of compact Riemannian manifolds with boundary  $M$ , such that there exists a neighborhood  $W$  of  $\partial M$  that is isometric to a product  $\partial M \times [0, \epsilon)$ .

When we have such a manifold with boundary  $M$ , the double of  $M$  which will be denoted by  $M'$  will be a smooth compact manifold without boundary equipped with an involution  $p \mapsto p^*$  that sends a point  $p$  to its reflection through the boundary.

The Dirichlet heat kernel on  $M$  is

$$(120) \quad K_t(x, y) = K'_t(x, y) - K'_t(x, y^*)$$

where  $K'_t(x, y)$  is the heat kernel on  $M'$ .

The existence of an asymptotic expansion of  $K'_t(x, y)$  implies that

$$(121) \quad K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y) t^i + e^{-d(x, y^*)^2/4t} \sum_i \psi_i(x, y) t^i$$

where  $\psi_i(x, y) = -\phi_i(x, y^*)$ .

This can be used to define  $f_\gamma^N[P, I]$  and to show that  $|f_\gamma[P, I] - f_\gamma^N[P, I]|$  is bounded by a power of  $t_k$  the increases linearly with  $N$ .

Choose a finite cover  $U_1, \dots, U_m$  of  $\partial M$  by coordinate neighborhoods. This induces a finite cover  $U_1 \times [0, \epsilon), \dots, U_m \times [0, \epsilon)$  of  $W \cong \partial M \times [0, \epsilon)$  by coordinate neighborhoods.

Then choose a finite cover  $V_1, \dots, V_{m'}$  of the complement of  $M \times [0, \epsilon)$ .

On the open sets  $U_i$  that intersect the boundary, since  $M$  is a product, we can use the Pythagorean theorem and the square distance becomes

$$(122) \quad d_M^2(x, y) = d_{\partial M}^2(\bar{x}, \bar{y}) + |x_n - y_n|^2$$

Therefore, on these open sets we can apply the analysis of 4.3 for the direction normal to the boundary and the analysis of 4.4 to the  $\partial M$  direction. On open sets  $V_i$  whose closures do not intersect the boundary,

$$(123) \quad K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y) t^i$$

so we can apply the analysis of 4.4.

## 5. CONSTRUCTION OF AN EFFECTIVE FIELD THEORY FROM A LOCAL FUNCTIONAL

In this section, we show that an effective action can be constructed from a local functional  $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$  using a procedure that is based on Theorem 8.

Do the following for each sequence  $I^{(1)}, \dots, I^{(p)}$  as in Corollary 3: Let  $N'_1$  be the smallest nonnegative integer such that  $d_1(N'_1) \geq 0$ . Let  $N'_2$  be the smallest nonnegative integer such that  $d_2(N'_1, N'_2) \geq 0$  and so on. Then by Theorem 8,

$$(124) \quad |f_\gamma[P, I](\mathbf{t}) - f_\gamma^{N'_1, \dots, N'_p}[P, I](\mathbf{t})| \leq C$$

for some constant  $C$ . Let  $\bar{E}_R^{I^{(1)}, \dots, I^{(p)}} = \bar{E}_R^{I^{(1)}} \cap \bar{E}_R^{I^{(2)}} \cdots \cap \bar{E}_R^{I^{(p)}}$ ,  
Let

$$(125) \quad w_\gamma^{\text{CT}}(P_\epsilon^1, I) = \sum_{p=1}^k \sum_{I^{(1)}, \dots, I^{(p)}} \int_{\bar{E}_R^{I^{(1)}, \dots, I^{(p)}}} f_\gamma^{N'_1, \dots, N'_p}[P, I](\mathbf{t}) dt$$

We can integrate this formula on  $(\epsilon, 1)^k \cap \bar{E}_R^{I^{(1)}, \dots, I^{(p)}}$ . This gives

$$(126) \quad |w_\gamma(P_\epsilon^1, I) - w_\gamma^{\text{CT}}(P_\epsilon^1, I)| \leq C(1 - \epsilon^k).$$

In particular, by Lebesgue's dominated convergence theorem, we can let  $\epsilon \rightarrow 0^+$ .

Thus, the limit as  $\epsilon \rightarrow 0^+$  of  $w_\gamma(P_\epsilon^L, I) - w_\gamma^{\text{CT}}(P_\epsilon^1, I)$  exists as well. We shall call this the renormalized Feynman weight.

The counterterms for the effective action are defined by

$$(127) \quad I_{i,k}^{\text{CT}}(\epsilon) = W_{i,k}^{\text{CT}} \left( P_\epsilon^1, I - \sum_{(i', k') \prec (i, k)} I_{i', k'}^{\text{CT}}(\epsilon) \right),$$

where

$$(128) \quad W_{i,k}^{\text{CT}}(P_\epsilon^1, I) = \sum_{\substack{\gamma \text{ conn} \\ g(\gamma)=i, T(\gamma)=k}} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)} w_\gamma^{\text{CT}} \left( P_\epsilon^1, I - \sum_{(i', k') \prec (i, k)} I_{i', k'}^{\text{CT}}(\epsilon) \right)$$

Then the effective action is defined by

$$(129) \quad I[L] = \lim_{\epsilon \rightarrow 0^+} W(P_\epsilon^L, I - I^{\text{CT}}(\epsilon)).$$

This is well-defined because for all  $i, k$ ,

$$(130) \quad I_{i,k}[L] = \lim_{\epsilon \rightarrow 0^+} W_{i,k}(P_\epsilon^L, I - I^{\text{CT}}(\epsilon))$$

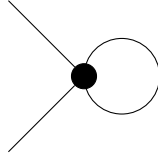
$$(131) \quad = \lim_{\epsilon \rightarrow 0^+} \left[ W_{i,k} \left( P_\epsilon^L, I - \sum_{(i',k') \prec (i,k)} I_{i',k'}^{\text{CT}}(\epsilon) \right) - I_{(i,k)}^{\text{CT}}(\epsilon) \right].$$

## 6. SOME EXAMPLES

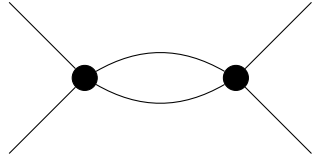
In this section, we shall compute  $w_\gamma^{\text{CT}}(P_\epsilon^1, I)$  on  $\mathbb{H}^4$  with the Euclidean metric for

$$(132) \quad I = c \frac{1}{4!} \int \phi^4$$

and  $\gamma = \gamma_1, \gamma_2$ , where  $\gamma_1$  is the graph



and  $\gamma_2$  is the graph



In the appendix of [1], Costello computes the counterterms on  $\mathbb{R}^4$  for these graphs.

In the first case, there is only one vertex and thus no need to apply the methods of Section 4.3:

$$(133) \quad w_\gamma^{\text{CT}}(P_\epsilon^1, I) = c \int_\epsilon^1 \int_{\mathbb{H}^4} K_t(x, x) \phi(x)^2$$

$$(134) \quad = c \int_\epsilon^1 \int_{\mathbb{H}^4} (4\pi t)^{-2} (1 - e^{-x_n^2/t}) \phi(x)^2$$

$$(135) \quad = c \frac{1}{16\pi^2} \left[ \left( \frac{1}{\epsilon} - 1 \right) \int_{\mathbb{H}^4} \phi(x)^2 - \int_\epsilon^1 \int_{\mathbb{H}^4} (4\pi t)^{-2} e^{-x_n^2/t} \phi(x)^2 \right]$$

$$(136) \quad \equiv c \frac{1}{16\pi^2} \left[ \frac{1}{\epsilon} \int_{\mathbb{H}^4} \phi(x)^2 - \int_\epsilon^1 \int_{\mathbb{H}^4} (4\pi t)^{-2} e^{-x_n^2/t} \phi(x)^2 \right]$$

where we use the symbol  $\equiv$  to signify that the existence of the limit  $w_\gamma(P_\epsilon^L, I) - w_\gamma^{\text{CT}}(P_\epsilon^1, I)$  is unaffected by adding functionals to  $w_\gamma^{\text{CT}}(P_\epsilon^1, I)$  that depend on  $\epsilon$  whose limit as  $\epsilon \rightarrow 0^+$  exists.

In the second case, there are in fact two vertices, and

(137)

$$w_\gamma(P_\epsilon^1, I) = c^2 \int_{[\epsilon, 1]^2} \int_{(\mathbb{H}^4)^2} K_{t_1}(x_1, x_2) K_{t_2}(x_1, x_2) \phi(x_1)^2 \phi(x_2)^2$$

(138)

$$= \int_{[\epsilon, 1]^2} [f_{\gamma, 0, 0}(t_1, t_2, \phi) + f_{\gamma, 1, 0}(t_1, t_2, \phi) + f_{\gamma, 0, 1}(t_1, t_2, \phi) + f_{\gamma, 1, 1}(t_1, t_2, \phi)]$$

where

(139)

$$f_{\gamma, 0, 0}(t_1, t_2, \phi) = c^2 \frac{1}{256\pi^4} \int_{(\mathbb{H}^4)^2} t_1^{-2} t_2^{-2} e^{-\|x_1 - x_2\|^2(1/4t_1 + 1/4t_2)} \phi(x_1)^2 \phi(x_2)^2$$

(140)

$$f_{\gamma, 0, 1}(t_1, t_2, \phi) = -c^2 \frac{1}{256\pi^4} \int_{(\mathbb{H}^4)^2} t_1^{-2} t_2^{-2} e^{-\|x_1 - x_2\|^2/4t_1 - \|x_1 - x_2^*\|^2/4t_2} \phi(x_1)^2 \phi(x_2)^2$$

(141)

$$f_{\gamma, 1, 0}(t_1, t_2, \phi) = -c^2 \frac{1}{256\pi^4} \int_{(\mathbb{H}^4)^2} t_1^{-2} t_2^{-2} e^{-\|x_1 - x_2^*\|^2/4t_1 - \|x_1 - x_2\|^2/4t_2} \phi(x_1)^2 \phi(x_2)^2$$

(142)

$$f_{\gamma, 1, 1}(t_1, t_2, \phi) = c^2 \frac{1}{256\pi^4} \int_{(\mathbb{H}^4)^2} t_1^{-2} t_2^{-2} e^{-\|x_1 - x_2^*\|^2/4t_1 - \|x_1 - x_2\|^2/4t_2} \phi(x_1)^2 \phi(x_2)^2.$$

For  $f_{\gamma, 0, 0}$ ,  $f_{\gamma, 1, 0}$ ,  $f_{\gamma, 0, 1}$ ,  $f_{\gamma, 1, 1}$ , introduce the coordinates,  $\bar{w} = \frac{\bar{x}_1 + \bar{x}_2}{2}$  and  $\bar{y} = \frac{\bar{x}_1 - \bar{x}_2}{2}$  and  $x_{1, n} = u + z$  and  $x_{2, n} = u - z$ . Note that the Jacobian of these transformations is  $2^4$ . Take the Taylor expansion to order 0 of  $\phi(x_1)^2 \phi(x_2)^2$  in  $\bar{y}$  and  $z$ .

On  $t_2^R \leq t_1$ , we have

(143)

$$f_{\gamma, 0, 0}^{N'}(t_1, t_2, \phi) = \frac{c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|\bar{y}\|^2 + |z|^2)(1/t_1 + 1/t_2)} P_{N'}(\phi)(\bar{w}, u)$$

where

$$(144) \quad P_{N'}(\phi)(\bar{w}, u) = \sum_{|I| + i \leq N'} \frac{1}{(|I| + i)!} \frac{\partial f}{\partial y^I \partial z^i}(\bar{w}, 0, u, 0) y^I z^i.$$

where  $f(\bar{w}, \bar{y}, u, z) = \phi^2(\bar{w} + \bar{y}, u + z) \phi^2(\bar{w} - \bar{y}, u - z)$ , and the error is bounded

$$(145) \quad |f_{\gamma, 0, 0}^{N'}(t_1, t_2, \phi) - f_{\gamma, 0, 0}^{N'}(t_1, t_2, \phi)| \leq C t_2^{\frac{1}{2}(N' + 1) + \frac{1}{2}4} t_2^{-2 - 2R} \leq C$$

in particular, for  $R = 3$  and  $N' = 11$ . The integral of the order 0 part of the Taylor expansion  $y$  is

$$(146) \quad \frac{c^2}{16\pi^4} t_1^{-2} t_2^{-2} \left( \frac{\pi}{1/t_1 + 1/t_2} \right)^{3/2} \int_{-u}^u e^{-z^2/t_2} dz \int_{\mathbb{H}^4} \phi(\bar{w}, u)^4.$$

The interested reader can compute the higher orders of  $f_{\gamma, 0, 0}^{N'}(t_1, t_2, \phi)$  using the formula for the Wick integral from 3.2.

For the other terms  $f_{\gamma,1,0}^{N'}$ ,  $f_{\gamma,0,1}^{N'}$ ,  $f_{\gamma,1,1}^{N'}$ , we have

(147)

$$f_{\gamma,1,0}^{N'}(t_1, t_2, \phi) = \frac{-c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|\bar{y}\|^2 + u^2)/t_1 - (\|\bar{y}\|^2 + z^2)/t_2} P_{N'}(\phi)(\bar{w}, u)$$

for which the integral of the order 0 part of the Taylor expansion  $\bar{y}$  and  $z$  is given by

$$(148) \quad \frac{c^2}{16\pi^4} \left( \frac{\pi}{1/t_1 + 1/t_2} \right)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} \left[ e^{-u^2/t_1} \int_{-u}^u e^{-z^2/t_2} \right] \phi(\bar{w}, u)^4.$$

And

(149)

$$f_{\gamma,0,1}^{N'}(t_1, t_2, \phi) = \frac{-c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|\bar{y}\|^2 + z^2)/t_1 - (\|\bar{y}\|^2 + u^2)/t_2} P_{N'}(\phi)(w, z)$$

for which the integral of the order 0 part of the Taylor expansion  $\bar{y}$  and  $z$  is given by

$$(150) \quad \frac{c^2}{16\pi^4} \left( \frac{\pi}{1/t_1 + 1/t_2} \right)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} \left[ e^{-u^2/t_2} \int_{-u}^u e^{-z^2/t_1} \right] \phi(\bar{w}, u)^4.$$

And, lastly

(151)

$$f_{\gamma,1,1}^{N'}(t_1, t_2, \phi) = \frac{c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|\bar{y}\|^2 + u^2)(1/t_1 + 1/t_2)} P_{N'}(\phi)(\bar{w}, z)$$

for which the integral of the order 0 part of the Taylor expansion  $\bar{y}$  and  $z$  is given by

$$(152) \quad \frac{c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|\bar{y}\|^2 + u^2)(1/t_1 + 1/t_2)} \phi(\bar{w}, u)^4 = 0$$

In each of these cases, the contributions from the parts of the Taylor polynomial of degree 1 and higher can be computed by the interested reader using the results of 3.2.

Lastly, on the set where  $t_1 \leq t_2^R$  we have

$$(153) \quad f_{\gamma,0,0}^{N'}(t_1, t_2, \phi) = \frac{c^2}{16\pi^4} \int_0^\infty \int_{-u}^u \int_{(\mathbb{R}^3)^2} t_1^{-2} t_2^{-2} e^{-(\|y\|^2 + z^2)/t_1} P_{N'}(\phi)(\bar{w}, u)$$

where

$$(154) \quad P_{N'}(\phi)(\bar{w}, u) = \sum_{|I|+i \leq N'} \frac{1}{(|I|+i)!} \frac{\partial f}{\partial \bar{y}^I z^i}(\bar{w}, 0, u, 0) y^I z^i.$$

where  $f(\bar{w}, \bar{y}, u, z) = e^{-(\|\bar{y}\|^2 + z^2)/t_2} \phi^2(\bar{w} + \bar{y}, u + z) \phi^2(\bar{w} - \bar{y}, u - z)$ . Thus to first order, that is for  $N' = 1$ ,

$$(155) \quad f_{\gamma',\gamma,0,0}^{N'}(t_1, t_2, \phi) = \frac{c^2}{16\pi^4} \left( \int_{-u}^u e^{-z^2/t_1} \right) (\pi t_1)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} \phi(\bar{w}, u)^4$$

and similarly to first order

$$(156) \quad f_{\gamma', \gamma, 0, 1}^{N'}(t_1, t_2, \phi) = \frac{-c^2}{16\pi^4} \left( \int_{-u}^u e^{-z^2/t_1} \right) (\pi t_1)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} e^{-u^2/t_2} \phi(\bar{w}, u)^4.$$

$$(157) \quad f_{\gamma', \gamma, 1, 0}^{N'}(t_1, t_2, \phi) = \frac{-c^2}{16\pi^4} (\pi t_1)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} 2ue^{-u^2/t_1} \phi(\bar{w}, u)^4$$

$$(158) \quad f_{\gamma', \gamma, 1, 1}^{N'}(t_1, t_2, \phi) = \frac{c^2}{16\pi^4} (\pi t_1)^{3/2} \frac{1}{t_1^2 t_2^2} \int_{\mathbb{H}^4} 2ue^{-u^2(1/t_1 + 1/t_2)} \phi(\bar{w}, u)^4$$

Having taken the Taylor expansion to order  $N' = 1$ , the error is bounded by

$$(159) \quad Ct_2^{-2-N'} t_1^{-2} t_1^{\frac{N'+1}{2} + \frac{4}{2}} = Ct_2^{R\frac{N'+1}{2} - 2 - N'} = Ct_2^{3-2-1} = C$$

as desired.

#### REFERENCES

- [1] Kevin Costello. *Renormalization and effective field theory*. Vol. 170. American Mathematical Society Providence, 2011.
- [2] L. P. Kadanoff. “Scaling laws for Ising models near T(c)”. In: *Physics* 2 (1966), pp. 263–272.
- [3] Peter Kopietz, Lorenz Bartosch, and Florian Schütz. *Introduction to the functional renormalization group*. Vol. 798. Springer, 2010.
- [4] Kenneth G Wilson. “Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture”. In: *Physical review B* 4.9 (1971), p. 3174.
- [5] Kenneth G Wilson. “The renormalization group: Critical phenomena and the Kondo problem”. In: *Reviews of Modern Physics* 47.4 (1975), p. 773.